McMaster University
Department of Computing and Software
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CAS 701
Fall 2007
Assignment 1

CAS 701 — Logic and Discrete Mathematics in Software Engineering

19 September 2007 — **due 26 September 2007**

Exercise 1.1

Let $f : A \leftrightarrow B$ and $g : B \leftrightarrow C$ be relations, and let $h : A \leftrightarrow C$ be their composition, $h := f ; g$.

(a) Prove that, if f and g are injective, then h is injective, but the converse is false.

(b) Prove that, if f and g are surjective, then h is surjective, but the converse is false.

Let $R : A \leftrightarrow B$ be a total relation and $S : A \nrightarrow B$ a univalent relation.

(c) Prove that, if $R \subseteq S$, then $R = S$.

Exercise 1.2

We define a **rough set** on a carrier A be a pair (P, C) where P (for *possibly*) and C (for *certainly*) are both subsets of A, and $C \subseteq P$.

We define the relation $\preceq$ on rough sets (on A) as follows:

$$(P_1, C_1) \preceq (P_2, C_2) \quad \text{iff} \quad P_1 \subseteq P_2 \quad \text{and} \quad C_1 \subseteq C_2.$$

(a) Show that $\preceq$ is an ordering.

(b) Show that $\preceq$ is a lattice ordering, and provide explicit definitions for join and meet in that lattice.

(c) Show the algebraic lattice axioms for your explicit definitions of join and meet.

(d) Is this lattice modular? Provide a proof for your answer.

(e) Is this lattice distributive? Provide a proof for your answer.

(f) Is this lattice complete? Provide a proof for your answer.

Given a surjective **approximation mapping** $\delta : S \to A$ from a *space* set S to the *approximation* carrier A, we say that a pair (P, C) of subsets of A is an **approximation** of a set $X \subseteq S$ iff for every $a : A$ we have:

$$\begin{array}{llllll} a \in P & \text{iff there is an} & x : S & \text{with} & \delta(x) = a & \text{such that} \quad x \in X; \\ a \in C & \text{iff for all} & x : S & \text{with} & \delta(x) = a & \text{we have} \quad x \in X. \end{array}$$

(g) Show that each approximation of $X \subseteq S$ is a rough set.

(h) Show that each set $X \subseteq S$ has exactly one approximation.

We write Δ to denote the mapping from subsets of S to their approximation via δ.

(i) Is Δ an order homomorphism? If yes, provide a proof; otherwise, provide a counterexample.

(j) Is Δ a lattice homomorphism? If yes, provide a proof; otherwise, provide a counterexample.

Note: Instead of starting from the projection mapping δ as we do here, the rough set literature starts from the equivalence relation $\delta ; \delta^{\smile}$.

Exercise 1.3

A **simple graph** is a pair (N, E) consisting of a set N of *nodes* and a relation $E : N \leftrightarrow N$, which can be considered as a set of *edges*. We define the **subgraph** ordering $\preceq$ on simple graphs as follows:

$$(P_1, C_1) \preceq (P_2, C_2) \quad \text{iff} \quad P_1 \subseteq P_2 \quad \text{and} \quad C_1 \subseteq C_2.$$

(a) What are the atoms in the resulting lattice? State your answer formally and provide a proof.

(b) What are the join-irreducible elements in the resulting lattice? State your answer formally and provide a proof.

(c) Which subgraphs have complements? State your answer formally and provide a proof.