McMaster University
Department of Computing and Software
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CAS 701
Fall 2007
Assignment 2

CAS 701 — Logic and Discrete Mathematics in Software Engineering

5 November 2007 — **due 19 November 2007**

Bonus 20% on parts handed in (in final version) by the 12th.

Note: When quantifiers are used in the " $\bullet$ " notation, the scope of the quantifier extends ***as far as syntactically possible***.

For example: $\forall x \bullet \phi \wedge \psi$ reads $\forall x \bullet (\phi \wedge \psi)$
$\forall x \bullet (\forall y \bullet \phi \wedge \psi) \rightarrow \eta$; reads $\forall x \bullet ((\forall y \bullet (\phi \wedge \psi)) \rightarrow \eta)$.

1 Natural Deduction Proofs in Predicate Logic

For each of the following sequents, produce a formal proof on paper, and transscribe your proof into the language of the "Logic Daemon" proof checker (http://logic.tamu.edu/daemon.html) and check your proofs. Using the derived rules provided by the "Logic Daemon" is permitted in both proofs.

Please send the ASCII **source** of your "Logic Daemon" proofs by e-mail to the instructor.

(a) $\exists x \bullet S \rightarrow Q(x) \ \vdash\ S \rightarrow \exists x \bullet Q(x)$

(b) $S \rightarrow \exists x \bullet Q(x) \ \vdash\ \exists x \bullet S \rightarrow Q(x)$

(c) $(\exists x \bullet P(x)) \rightarrow S \ \vdash\ \forall x \bullet P(x) \rightarrow S$

(d) $S \rightarrow \forall x \bullet Q(x) \ \vdash\ \forall x \bullet S \rightarrow Q(x)$

(e) $\forall x \bullet S \rightarrow Q(x) \ \vdash\ S \rightarrow \forall x \bullet Q(x)$

(f) $\exists x \bullet Q(x) \rightarrow S \ \vdash\ (\forall x \bullet Q(x)) \rightarrow S$

(g) $(\forall x \bullet Q(x)) \rightarrow S \ \vdash\ \exists x \bullet Q(x) \rightarrow S$ **(not easy)**

2 Counter-Models

Assuming that our proof calculus for predicate logic is sound, show, by providing countermodels, that the validity of the following sequents cannot be proved. I.e., for each sequent provide a structure such that all formulae to the left of $\vdash$ are satisfied (evaluate to True) and the sole formula to the right of $\vdash$ is not satisfied.

(a) $\forall x \bullet P(x) \vee Q(x) \ \vdash\ (\forall x \bullet P(x)) \vee \forall x \bullet Q(x)$

(b) $(\forall x \bullet P(x)) \rightarrow S \ \vdash\ \forall x \bullet P(x) \rightarrow S$

(c) $\forall x \bullet \exists y \bullet R(x, y) \ \vdash\ \exists x \bullet \forall y \bullet R(x, y)$

3 Graphs as Structures

Let L_{Graph} be the language having two sorts N and E, and two unary function symbols $\mathsf{src}, \mathsf{trg} : E \to N$, and equality. A **graph** can now be defined as a structure for L_{Graph}. Which graphs satisfy the following formulae? For each of the following items, give a precise general natural-language description of the graphs satisfying the given formula, and draw three of these graphs that are "as different as possible":

(a) $\forall e_1 : E \bullet \forall e_2 : E \bullet e_1 = e_2$

(b) $\exists n : N \bullet \forall e : E \bullet n = \mathsf{src}(e)$

(c) $\forall e_1 : E \bullet \exists e_2 : E \bullet \mathsf{trg}(e_1) = \mathsf{src}(e_2) \wedge \neg(e_1 = e_2)$

Give a predicate logic formula (if possible) that is satisfied exactly in

(a) graphs without sources

(b) graphs with exactly one source

(c) directed forests

(d) directed trees

(e) acyclic graphs

(f) simple cycles

(g) cliques

(h) graphs without three-cliques

4 Theory of Partial Orders and Lattices

(a) State a finite set of axioms over an appropriate single-sorted signature that ensure that exactly partially ordered sets are models.

(b) Extend your solution conservatively with axioms characterising the ternary predicate symbol isLub such that for each model A the interpretation isLub^A contains the triple (j, x, y) if and only if j is the least upper bound in A of x and y.

(c) State and prove formally that the least upper bound is uniquely determined (if it exists).