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CAS 701
Fall 2007
Exercise Sheet 2

CAS 701 — Logic and Discrete Mathematics in Software Engineering

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1 Lattice Basics

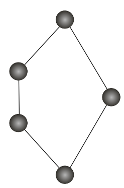
Lattices can be defined as ordered sets of the form $(L, \leq)$ or as algebras of the form $(L, \vee, \wedge)$. Work out the details to show that these two ways to define the notion of lattice are equivalent.

2 Sublattices

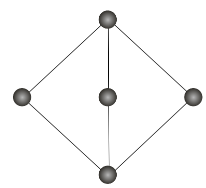
(a) Define “sublattice”.

(b) List all ***different*** (i.e., non-isomorphic) sublattices of M_3.

(c) List all ***different*** sublattices of N_5.



N_5



M_3

3 Distributive Lattices

Let L be a lattice. Prove that the following are equivalent:

(a) The equation $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds in L.

(b) The equation $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ holds in L.

(c) The equation $(x \vee y) \wedge (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ holds in L.

(d) L has no sublattice isomorphic with N_5 and no sublattice isomorphic with M_3.

4 Join-irreducibility [Burris-Sanka. 1.1.10]

If L is a finite lattice, show that every element is of the form $a_1 \vee \ldots \vee a_n$ where each a_i is join-irreducible.

5 Ideals [Burris-Sanka. 1.2.5 and 1.3.2]

If L is a lattice, then

- a **lower segment** of L is a ***downward-closed*** subset $S \subseteq L$, i.e., whenever $s \in S$ and $x \in L$ with $x \leq s$, then $x \in S$;
- an **ideal** of L is a nonempty lower segment that is closed under $\vee$.

Show that the set $I(L)$ of ideals of L forms a lattice under with the ordering $\subseteq$. Show that, if L is distributive, then the lattice $(I(A), \subseteq)$ is distributive, too.