Collagories for Relational Adhesive Rewriting

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Abstract

We define collagories essentially as "distributive allegories without zero morphisms", and show that they are sufficient for accommodating the relation-algebraic approach to graph transformation. Collagories closely correspond to the adhesive categories important for the categorical DPO approach to graph transformation. but thanks to their relation-algebraic flavour provide a more accessible and more flexible setting.

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Contents

1 Introduction 3

2 Categories, Allegories 3

3 Collagories 5

4 Tabulations and Co-tabulations 6

5 Maps in Collagories form Adhesive Categories 11

6 Collagories of Semi-Unary Algebras and Bisimulations 14

7 Reducts Along Signature Homomorphisms 20

8 Reduct-Restricted Σ-Algebra Categories 21

9 Conclusion 23

1 Introduction

One of the hallmarks of the relation-algebraic approach to graph transformation [Kaw90, Kah01, Kah04] is that it allows an abstract characterisation of the gluing condition for the double pushout approach. Nevertheless, the categorical approach to graph transformation has continued to use the node-and-edge-based formulation of the gluing condition even in the handbook chapter [CMR+97]. Recently, the literature of the categorical approach, starting essentially with [EPPH06] has adopted the “adhesive categories” of Lack and Sobociński [LS04], where however the details of the gluing condition are completely sidestepped.

Although the toposes of graph structures that give rise to the relational categories used in the relational approach are examples of adhesive categories, the latter also include, for example, categories of pointed set, which do not give rise to distributive allegories due to the failure of the zero law.

In this paper we show that dropping the zero law still produces a relational formalism that can accommodate the necessary tools for graph transformation, and furthermore relates nicely with adhesive categories.

We first re-develop, in sections 2–4, the fundamentals of the relation-algebraic approach to graph transformation using our new *bi-tabular collagories*, and show in Sect. 5 that these provide adhesive categories. Sections 6–8 are then devoted to constructing concrete bi-tabular collagories of algebras.

A shorter version of this work appears as [Kah09].

2 Categories, Allegories

This section only serves to fix notation and terminology for standard concepts, see [FS90, SS93, Kah04]. Like Freyd and Scedrov and a slowly increasing number of categorists, we use denote composition in “diagram order” not only in relation-algebraic contexts, where this is customary, but also in the context of categories. We will always use the infix operator “;” to make composition explicit: $R ; S = \mathcal{A} \xrightarrow{R} \mathcal{B} \xrightarrow{S} \mathcal{C}$.

Definition 2.1 A *category* $\mathbf{C}$ is a tuple $(\mathsf{Obj}_{\mathbf{C}}, \mathsf{Mor}_{\mathbf{C}}, \mathsf{src}, \mathsf{trg}, \mathbb{I}, ;)$ where

- $\mathsf{Obj}_{\mathbf{C}}$ is a collection of *objects*.
- $\mathsf{Mor}_{\mathbf{C}}$ is a collection of *arrows* or *morphisms*.
- src (resp. trg) maps each morphism to its source (resp. target) object.
  Instead of $\mathsf{src}(f) = \mathcal{A} \wedge \mathsf{trg}(f) = \mathcal{B}$ we write $f : \mathcal{A} \to \mathcal{B}$.
  The collection of all morphisms f with $f : \mathcal{A} \to \mathcal{B}$ is denoted as $\mathsf{Mor}_{\mathbf{C}}[\mathcal{A}, \mathcal{B}]$ and also called a *homset*.
- “;” is the binary *composition* operator, and composition of two morphisms $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B}' \to \mathcal{C}$ is defined iff $\mathcal{B} = \mathcal{B}'$, and then $(f ; g) : \mathcal{A} \to \mathcal{C}$; composition is associative.
- $\mathbb{I}$ associates with every object $\mathcal{A}$ a morphism $\mathbb{I}_{\mathcal{A}}$ which is both a right and left unit for composition. □

Definition 2.2 An *ordered category* is a category $\mathbf{C}$ such that

- for each two objects $\mathcal{A}$ and $\mathcal{B}$, the relation $\sqsubseteq_{\mathcal{A},\mathcal{B}}$ is a partial order on $\mathsf{Mor}_{\mathbf{C}}[\mathcal{A},\mathcal{B}]$ (the indices will usually be omitted), and
- composition is monotonic with respect to $\sqsubseteq$ in both arguments. □

Definition 2.3 An *upper-semilattice category* is an ordered category where

- each homset is a distributive lattice with binary join $\sqcup$,
- composition distributes over binary joins from both sides. □

For homsets that have least or greatest elements, we introduce corresponding notation:

Definition 2.4 In an ordered category, for each two objects $\mathcal{A}$ and $\mathcal{B}$ we introduce the following notions:

- If the homset $\mathsf{Mor}_{\mathbf{C}}[\mathcal{A},\mathcal{B}]$ contains a greatest element, this is denoted $\mathbb{T}_{\mathcal{A},\mathcal{B}}$.
- If the homset $\mathsf{Mor}_{\mathbf{C}}[\mathcal{A},\mathcal{B}]$ contains a least element, this is denoted $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}}$. □

For these extremal morphisms and for identities we frequently omit indices where these can be induced from the context.

Definition 2.5 An *allegory* is an ordered category such that

- each morphism $R : \mathcal{A} \to \mathcal{B}$ has a *converse* $R^{\smile} : \mathcal{B} \to \mathcal{A}$,
- the *involution equations* hold for all $R : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{C}$:

$$(R^{\smile})^{\smile} = R \qquad \mathbb{I}_{\mathcal{A}}^{\smile} = \mathbb{I}_{\mathcal{A}} \qquad (R\,;S)^{\smile} = S^{\smile}\,;R^{\smile}$$

- conversion is monotonic with respect to $\sqsubseteq$.
- each homset is a lower semilattice with binary meet $\sqcap$.
- for all $Q : \mathcal{A} \to \mathcal{B}$, $R : \mathcal{B} \to \mathcal{C}$, and $S : \mathcal{A} \to \mathcal{C}$, the *modal rule* holds:

$$Q\,;R \sqcap S \sqsubseteq (Q \sqcap S\,;R^{\smile})\,;R \ .$$

□

Many standard properties of relations can be characterised in the context of allegories:

Definition 2.6 A morphism $R : \mathcal{A} \to \mathcal{B}$ in an allegory is called:

- *univalent* iff $R^{\smile}\,;R \sqsubseteq \mathbb{I}_{\mathcal{B}}$,
- *total* iff $\mathbb{I}_{\mathcal{A}} \sqsubseteq R\,;R^{\smile}$,
- *injective* iff $R\,;R^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{A}}$,
- *surjective* iff $\mathbb{I}_{\mathcal{B}} \sqsubseteq R^{\smile}\,;R$,
- a *mapping* iff it is univalent and total,
- *bijective* iff it is injective and surjective,
- *difunctional* iff $R\,;R^{\smile}\,;R \sqsubseteq R$. (See [SS93, 4.4] for more about difunctionality). □

For an allegory $\mathbf{A}$, we write $\mathsf{Map}\,\mathbf{A}$ for the sub-category of $\mathbf{A}$ that contains only the mappings as arrows.

Definition 2.7 For a morphism $R : \mathcal{A} \to \mathcal{B}$ in an allegory, we define its *difunctional closure* $R^{\boxast} : \mathcal{A} \to \mathcal{B}$ as the least difunctional morphism containing R (if this exists), and we further define $R^{\triangleright} : \mathcal{A} \to \mathcal{A}$ and $R^{\triangleleft} : \mathcal{B} \to \mathcal{B}$ as:

$$R^{\triangleright} := \mathbb{I} \sqcup R^{\boxast} ; (R^{\boxast})^{\smile} \qquad \text{and} \qquad R^{\triangleleft} := \mathbb{I} \sqcup (R^{\boxast})^{\smile} ; R^{\boxast} \ . \qquad \square$$

For endomorphisms, there are a few additional properties of interest:

Definition 2.8 A morphism $R : \mathcal{A} \to \mathcal{A}$ in an allegory is called:

- *reflexive* iff $\mathbb{I} \sqsubseteq R$,
- *transitive* iff $R ; R \sqsubseteq R$,
- *co-reflexive* or a *sub-identity* iff $R \sqsubseteq \mathbb{I}_{\mathcal{A}}$,
- *symmetric* iff $R^{\smile} \sqsubseteq R$,
- an *equivalence* iff it is symmetric, reflexive and transitive. □

Definition 2.9 [FS90, 2.15] An object $\mathcal{U}$ in an allegory is a *partial unit* if $\mathbb{I}_{\mathcal{U}} = \mathbb{T}_{\mathcal{U},\mathcal{U}}$. The object $\mathcal{U}$ is a *unit* if, further, every object is the source of a total morphism targeted at $\mathcal{U}$. An allegory is said to be *unitary* if it has a unit. □

We use the symbol "$\mathbb{1}$" for an arbitrary but fixed unit object.

3 Collagories

$\kappa\acute{o}\lambda\lambda\alpha$: glue

In Freyd and Scedrov's treatment, although allegories do not require zero-ary meets, distributive allegories do require zero-ary joins (least elements) together with distributivity of composition over them, that is, the zero law $\perp\!\!\!\perp ; R = \perp\!\!\!\perp$. We define an intermediate concept that does not assume anything about zero-ary joins:

Definition 3.1 A *collagory* is an allegory that is also an upper-semilattice category. □

For Kleene star, we use Kozen's axioms [Koz94]:

Definition 3.2 A *Kleene collagory* is a collagory where, on homsets of endomorphisms, there is an additional unary operation $\_^{*}$ which satisfies the following axioms for all $R : \mathcal{A} \to \mathcal{A}$, $Q : \mathcal{B} \to \mathcal{A}$, and $S : \mathcal{A} \to \mathcal{C}$:

$$\begin{array}{rcccrcll}
 & & & & R^{*} & = & \mathbb{I}_{A} \sqcup R \sqcup R^{*} ; R^{*} & \text{recursive star definition} \\
Q ; R & \sqsubseteq & Q & \Rightarrow & Q ; R^{*} & \sqsubseteq & Q & \text{right induction} \\
R ; S & \sqsubseteq & S & \Rightarrow & R^{*} ; S & \sqsubseteq & S & \text{left induction}
\end{array} \qquad \square$$

Proposition 3.3 In a Kleene collagory, all difunctional closures exist, and:

$$R^{\rhd} = (R \mathbin{;} R^{\smile})^{*} \ , \qquad R^{\lhd} = (R^{\smile} \mathbin{;} R)^{*} \ , \qquad R^{\boxast} = R^{\rhd} \mathbin{;} R = R \mathbin{;} R^{\lhd} \ . \qquad \square$$

Alternatively, we also can fore-go the Kleene star and directly axiomatise difunctional closure:

Definition 3.4 A *difunctionally closed collagory* is a collagory where, there is an additional unary operation $\_^{\boxast}$ which satisfies the following axioms for all $R : \mathcal{A} \to \mathcal{B}$, $Q : \mathcal{C} \to \mathcal{A}$, and $S : \mathcal{A} \to \mathcal{C}$: $Q' : \mathcal{C} \to \mathcal{B}$, and $S' : \mathcal{B} \to \mathcal{C}$:

$$\begin{array}{rclcrclcrcll}
 & & & & R^{\boxast} & = & R \sqcup R^{\boxast} \mathbin{;} (R^{\boxast})^{\smile} \mathbin{;} R^{\boxast} & & & & & \text{recursive definition} \\
Q \mathbin{;} R & \sqsubseteq & Q' & \wedge & Q' \mathbin{;} R^{\smile} \mathbin{;} R & \sqsubseteq & Q' & \Rightarrow & Q \mathbin{;} R^{\boxast} & \sqsubseteq & Q' & \text{right induction} \\
R \mathbin{;} S & \sqsubseteq & S' & \wedge & R \mathbin{;} R^{\smile} \mathbin{;} S' & \sqsubseteq & S' & \Rightarrow & R^{\boxast} \mathbin{;} S & \sqsubseteq & S' & \text{left induction} \qquad \square
\end{array}$$

Proposition 3.5 In a difunctionally closed collagory, the operation $\_^{\boxast}$ produces difunctional closures.

Proof: Containment $R \sqsubseteq R^{\boxast}$ and difunctionality $R^{\boxast} \mathbin{;} (R^{\boxast})^{\smile} \mathbin{;} R^{\boxast} \sqsubseteq R^{\boxast}$ follow directly from the recursive definition.

For minimality, assume that C is difunctional with $R \sqsubseteq C$. Then we have $\mathbb{I} \mathbin{;} R \sqsubseteq C$ and $C \mathbin{;} R^{\smile} \mathbin{;} R \sqsubseteq C \mathbin{;} C^{\smile} \mathbin{;} C \sqsubseteq C$ and therefore, with the right induction rule, $R^{\boxast} = \mathbb{I} \mathbin{;} R^{\boxast} \sqsubseteq C$. $\square$

4 Tabulations and Co-tabulations

Central to the connection between pullbacks and pushouts in categories of mappings on the one hand and constructions in relational theories on the other hand is the fact that a square of mappings commutes iff the "relation" induced by the source span is contained in that induced by the target co-span. The proof of this does not need the modal rule.

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl, "P"'] \arrow[dr, "Q"] & \\
\mathcal{B} \arrow[dr, "R"] & & \mathcal{C} \arrow[dl, "S"'] \\
 & \mathcal{D} &
\end{tikzcd}$$

Lemma 4.1 [FS90, 2.146] Given a square of mappings in an allegory as drawn above, we have $P \mathbin{;} R = Q \mathbin{;} S$ iff $P^{\smile} \mathbin{;} Q \sqsubseteq R \mathbin{;} S^{\smile}$. $\square$

This provides a first hint that in the relational setting, the identity of the two mappings P and Q does not matter when looking for a pushout of the span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ — we only need to consider the diagonal $P^{\smile} \mathbin{;} Q$. Dually, when looking for a pullback of the co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$, only $R ; S^{\smile}$ needs to be considered. The gap between the two ways of calculating the horizontal diagonal can be significant since $R ; S^{\smile}$ is always difunctional.

Producing the result span of a pullback (respectively the result co-span of a pushout) from the horizontal diagonal alone is, in some sense, a generalisation of Freyd and Scedrov's splitting of idempotents; [Kah04] contains more discussion of this aspect.

Definition 4.2 [FS90, 2.14] In an allegory, let a morphism $V : \mathcal{B} \to \mathcal{C}$ be given. The span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ of *mappings* P and Q is called a *tabulation of* V iff the following equations hold:

$$P^{\smile} ; Q = V \qquad\qquad P ; P^{\smile} \sqcap Q ; Q^{\smile} = \mathbb{I}_{\mathcal{A}} \ . \qquad \square$$

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "P"'] \arrow[dr, "Q"] & \\ \mathcal{B} \arrow[rr, "V"] & & \mathcal{C} \end{tikzcd}$$

The following equivalent characterisation provided by [Kah04] has the advantage that it is fully equational, without the implicit inclusion conditions in the requirement that P and Q are mappings. This frequently facilitates calculations. Notice that $\mathbb{I} \sqcap V ; V^{\smile} = \mathsf{dom}\, V$; we use the expanded form to emphasise the duality with Prop. 4.6 below.

Proposition 4.3 In an allegory, the span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation of $V : \mathcal{B} \to \mathcal{C}$ if and only if the following equations hold:

$$P^{\smile} ; Q = V \qquad \begin{array}{rcl} P^{\smile} ; P & = & \mathbb{I} \sqcap V ; V^{\smile} \\ Q^{\smile} ; Q & = & \mathbb{I} \sqcap V^{\smile} ; V \end{array} \qquad P ; P^{\smile} \sqcap Q ; Q^{\smile} = \mathbb{I}_{\mathcal{A}} \ . \qquad \square$$

Tabulations in an allegory are unique up to isomorphism (this uses the modal rule), and include the following special cases:

- In a tabulation of a sub-identity, both tabulation morphisms are the induced *sub-object* injection [FS90, 2.145].
- We can define a *direct product* of $\mathcal{A}$ and $\mathcal{B}$ to be a tabulation of a $\mathbb{T}_{\mathcal{A},\mathcal{B}}$, provided that greatest morphism exists.
- If a co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ of mappings is given, then its *pullback* in $\mathsf{Map}\,\mathbf{A}$ is obtained as a tabulation of $R ; S^{\smile}$ [FS90, 2.147].

If an allegory in known to have all direct products and subobjects, then these can be used to construct a tabulation for each morphism.

Lemma 4.4 If a co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ of mappings is given with R injective, and $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation for $R ; S^{\smile}$, then Q is injective, too.

PROOF: First we use Prop. 4.3 to show $Q ; Q^{\smile} \sqsubseteq P ; P^{\smile}$:

$$\begin{array}{rcl} Q ; Q^{\smile} & = & Q ; Q^{\smile} ; Q ; Q^{\smile} \ = \ Q ; (\mathbb{I} \sqcap (R ; S^{\smile})^{\smile} ; R ; S^{\smile}) ; Q^{\smile} \\ & \sqsubseteq & Q ; S ; R^{\smile} ; R ; S^{\smile} ; Q^{\smile} \ = \ P ; R ; R^{\smile} ; R ; R^{\smile} ; P^{\smile} \ = \ P ; P^{\smile} \end{array}$$

Together with the tabulation condition, this implies $Q ; Q^{\smile} = P ; P^{\smile} \sqcap Q ; Q^{\smile} = \mathbb{I}_{\mathcal{A}}$, that is, injectivity of Q. $\square$

While a tabulation can be seen as a certain kind of decomposition of an arbitrary morphism in an allegory into a span, the dual of a tabulation is then a certain kind of decomposition of a difunctional morphism in a collagory into a co-span. Although the formal material here is dual to that above, we still spell it out in full detail for reference and better intuition.

Definition 4.5 [Kah04] In a collagory, let a morphism $W : \mathcal{B} \to \mathcal{C}$ be given. The co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ of *mappings* R and S is called a *co-tabulation of* W iff the following equations hold:

$$R ; S^{\smile} = W \qquad\qquad R^{\smile} ; R \sqcup S^{\smile} ; S = \mathbb{I}_{\mathcal{D}} \ . \qquad \square$$

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "W"] \arrow[dr, "R"] & & \mathcal{C} \arrow[dl, "S"'] \\ & \mathcal{D} & \end{tikzcd}$$

The first equation implies $W ; W^{\smile} ; W = R ; S^{\smile} ; S ; R^{\smile} ; R ; S^{\smile} \sqsubseteq R ; S^{\smile} = W$ (using univalence of R and S), so if W has a co-tabulation, it has to be difunctional.

Co-tabulations also have an equivalent characterisation that does not involve the mapping concept explicitly and is perfectly "bi-dual" to the tabulation characterisation in Prop. 4.3:

Proposition 4.6 In a collagory, the span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation of $W : \mathcal{B} \to \mathcal{C}$ iff the following equations hold:

$$R ; S^{\smile} = W \qquad \begin{array}{rcl} R ; R^{\smile} & = & \mathbb{I} \sqcup W ; W^{\smile} \\ S ; S^{\smile} & = & \mathbb{I} \sqcup W^{\smile} ; W \end{array} \qquad R^{\smile} ; R \sqcup S^{\smile} ; S = \mathbb{I}_{\mathcal{D}} \ . \qquad \square$$

In a collagory, co-tabulations are unique up to isomorphism [SS93, 4.4.10], and we have the following special cases:

- In a co-tabulation of an equivalence, both R and S are the induced *quotient* projections.
- We can define a *direct sum* of $\mathcal{A}$ and $\mathcal{B}$ to be a co-tabulation of $\perp\!\!\!\perp_{\mathcal{A},\mathcal{B}}$, if that least morphism exists.

If direct sums and quotients are available, then a co-tabulation can be constructed for each difunctional morphism.

To establish the relationship between the relation-algebraic co-tabulation definition and the universal characterisation of pushouts in categories, we first establish a generalised factorisation property for co-tabulations:

Lemma 4.7 In a collagory, let $W : \mathcal{B} \leftrightarrow \mathcal{C}$ be a difunctional morphism.

If the cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation of W, and if the cospan $\mathcal{B} \xrightarrow{R'} \mathcal{D}' \xleftarrow{S'} \mathcal{C}$ consists of mappings that satisfy

$$W ; S' \sqsubseteq R' \qquad \text{and} \qquad W^{\smile} ; R' \sqsubseteq S' \ ,$$

then $U : \mathcal{D} \to \mathcal{D}'$ with $U := R^{\smile} ; R' \sqcup S^{\smile} ; S'$ is a mapping such that $R' = R ; U$ and $S' = S ; U$.

PROOF: Factorisation follows easily from the assumptions:

$$\begin{array}{rcccccc} R ; U & = & R ; R^\smile ; R' \sqcup R ; S^\smile ; S' & = & (\mathbb{I} \sqcup W ; W^\smile) ; R' \sqcup W ; S' & = & R' \\ S ; U & = & S ; R^\smile ; R' \sqcup S ; S^\smile ; S' & = & W^\smile ; R' \sqcup (\mathbb{I} \sqcup W^\smile ; W) ; S' & = & S' \end{array}$$

Univalence follows from factorisation and univalence of R' and S':

$$U^\smile ; U = (R'^\smile ; R \sqcup S'^\smile ; S) ; U = R'^\smile ; R ; U \sqcup S'^\smile ; S ; U = R'^\smile ; R' \sqcup S'^\smile ; S' \sqsubseteq \mathbb{I}$$

Totality of U uses totality of R' and S', and the last co-tabulation condition:

$$\begin{array}{rcll} U ; U^\smile & \sqsupseteq & R^\smile ; R' ; R'^\smile ; R \sqcup S^\smile ; S' ; S'^\smile ; S & \text{Definition of } U \\ & \sqsupseteq & R^\smile ; R \sqcup S^\smile ; S & \text{Totality of } R' \text{ and } S' \\ & \sqsupseteq & \mathbb{I} & \text{R, S cotabulation} \end{array}$$

□

This helps to show that co-tabulations are unique up to isomorphism:

Theorem 4.8 In a collagory, let $W : \mathcal{B} \leftrightarrow \mathcal{C}$ be a difunctional morphism.

If the cospans $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ and $\mathcal{B} \xrightarrow{R'} \mathcal{D}' \xleftarrow{S'} \mathcal{C}$ are both co-tabulations for W, then there is a bijective mapping $U : \mathcal{D} \to \mathcal{D}'$ such that $R' = R ; U$ and $S' = S ; U$.

PROOF: With the co-tabulation conditions for $\mathcal{B} \xrightarrow{R'} \mathcal{D}' \xleftarrow{S'} \mathcal{C}$ and univalence of R' and S' we obtain:

$$W ; S' = R' ; S'^\smile ; S' \sqsubseteq R' \qquad \text{and} \qquad W^\smile ; R' = S' ; R'^\smile ; R' \sqsubseteq S' \ .$$

With Lemma 4.7 we know that $U := R^\smile ; R' \sqcup S^\smile ; S'$ is a mapping that factorises R' and S'.

By the same argument for $U^\smile$, we obtain that U is also bijective. □

A co-tabulation for a difunctional closure $Z^{\circledast}$ satisfies the following equations:

$$R ; S^\smile = Z^{\circledast} \qquad R ; R^\smile = Z^{\rhd} \qquad S ; S^\smile = Z^{\lhd} \qquad R^\smile ; R \sqcup S^\smile ; S = \mathbb{I}_{\mathcal{D}} \ .$$

This was introduced as a *gluing for U* in [Kah01]. Kawahara is the first to have characterised pushouts relation-algebraically in essentially this way [Kaw90]; he used relation-algebraic operations on relations arising in toposes.

The proof that this characterisation produces pushouts also in collagories is easily adapted from the proof of [Kah01, Theorem 5.3.5]:

Theorem 4.9 Let $\mathbf{C}$ be a collagory, and let $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ be a span in $\mathsf{Map}\,\mathbf{C}$, that is, P and Q are mappings.

If the cospan $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ in the collagory $\mathbf{C}$ is a co-tabulation for $W := (P^\smile ; Q)^{\circledast}$, then it is a pushout for $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ in $\mathsf{Map}\,\mathbf{C}$.

PROOF: The co-tabulation properties imply that R and S are mappings. For commutativity, we first show one inclusion:

$$P ; R \sqsupseteq P ; R ; \operatorname{ran} S = P ; R ; S^\smile ; S = P ; (P^\smile ; Q)^{\circledast} ; S \sqsupseteq P ; P^\smile ; Q ; S \sqsupseteq Q ; S$$

The opposite inclusion is derived in the same way, so we have equality.

Now assume another cospan $\mathcal{B} \xrightarrow{R'} \mathcal{D}' \xleftarrow{S'} \mathcal{C}$ in $\mathsf{Map}\,\mathbf{C}$ such that $P ; R' = Q ; S'$. This commutativity together with univalence of P and Q implies

$$P^{\smile} ; Q ; S' = P^{\smile} ; P ; R' \sqsubseteq R' \qquad \text{and} \qquad Q^{\smile} ; P ; R' = Q^{\smile} ; Q ; S' \sqsubseteq S' \ .$$

Using left induction for difunctional closure, this gives us:

$$W ; S' = (P^{\smile} ; Q)^{\boxast} ; S' \sqsubseteq R' \qquad \text{and} \qquad W^{\smile} ; R' = (Q^{\smile} ; P)^{\boxast} ; R' \sqsubseteq S' \ .$$

With Lemma 4.7 we then know that $U := R^{\smile} ; R' \sqcup S^{\smile} ; S'$ is a mapping that factorises R' and S'. So we only have to show that U is uniquely determined. Assume $U' : \mathcal{D} \to \mathcal{D}'$ with $R ; U' = R'$ and $S ; U' = S'$. Then:

$$U' = (R^{\smile} ; R \sqcup S^{\smile} ; S) ; U' = R^{\smile} ; R ; U' \sqcup S^{\smile} ; S ; U' = R^{\smile} ; R' \sqcup S^{\smile} ; S' = U \qquad \square$$

For pushouts along injective mappings, the difunctional closure becomes trivial:

Lemma 4.10 If a span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ of mappings is given with Q injective, then $P^{\smile} ; Q$ is difunctional (and therefore $(P^{\smile} ; Q)^{\boxast} = P^{\smile} ; Q$).

Proof: Since P, as a mapping, is difunctional, we have

$$P^{\smile} ; Q ; Q^{\smile} ; P ; P^{\smile} ; Q = P^{\smile} ; P ; P^{\smile} ; Q = P^{\smile} ; Q \ . \qquad \square$$

Furthermore, co-tabulations preserve injectivity:

Lemma 4.11 If a span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ of mappings is given with Q injective, and $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation for $P^{\smile} ; Q$, then R is injective, too.

Proof: Using injectivity of Q and univalence of P in one of the equations from Prop. 4.6 gives us injectivity of R:

$$R ; R^{\smile} = \mathbb{I} \sqcup P^{\smile} ; Q ; (P^{\smile} ; Q)^{\smile} = \mathbb{I} \sqcup P^{\smile} ; Q ; Q^{\smile} ; P = \mathbb{I} \sqcup P^{\smile} ; P = \mathbb{I} \ . \qquad \square$$

With that, we can show that, essentially, a pushout over an injective mapping is also a pullback:

Lemma 4.12 If a span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ of mappings is given with Q injective, and $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation for $P^{\smile} ; Q$, then $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is also a tabulation for $R ; S^{\smile}$.

Proof: Cross-commutativity $R ; S^{\smile} = P^{\smile} ; Q$ is already contained in the co-tabulation conditions. Since Q is injective and P is total, we also obtain

$$P ; P^{\smile} \sqcap Q ; Q^{\smile} = P ; P^{\smile} \sqcap \mathbb{I}_{\mathcal{A}} = \mathbb{I}_{\mathcal{A}} \ . \qquad \square$$

Definition 4.13 If a collagory has a tabulation for each morphism and a co-tabulation for each difunctional morphism, then we call it *bi-tabular*. $\square$

5 Maps in Collagories form Adhesive Categories

Adhesive categories as a more specific setting for double-pushout graph rewriting have been introduced by Lack and Sobociński [LS04, LS05]; the following two definitions are taken from there:

Definition 5.1 A *van Kampen square* (i) is a pushout which satisfies the following condition: given a commutative cube (ii) of which (i) forms the bottom face and the back faces are pullbacks (where $\mathcal{C}$ is considered to be in the back), the front faces are pullbacks if and only if the top face is a pushout.

$$\begin{tikzcd}
& \mathcal{C} \arrow[dl, "M"'] \arrow[dr, "F"] & \\
\mathcal{A} \arrow[dr, "G"'] & & \mathcal{B} \arrow[dl, "N"] \\
& \mathcal{D} &
\end{tikzcd}$$

(i)

$$\begin{tikzcd}
& \mathcal{C}' \arrow[rr, "f"] \arrow[dl, "m"'] \arrow[dd, "c" near end] & & \mathcal{B}' \arrow[dl, "n"'] \arrow[dd, "b"] \\
\mathcal{A}' \arrow[rr, "g" near end] \arrow[dd, "a"'] & & \mathcal{D}' \arrow[dd, "d" near start] & \\
& \mathcal{C} \arrow[rr, "F" near start] \arrow[dl, "M"] & & \mathcal{B} \arrow[dl, "N"] \\
\mathcal{A} \arrow[rr, "G"'] & & \mathcal{D} &
\end{tikzcd}$$

(ii) □

Definition 5.2 A category $\mathbf{C}$ is said to be *adhesive* if

1. $\mathbf{C}$ has pushouts along monomorphisms;
2. $\mathbf{C}$ has pullbacks;
3. pushouts along monomorphisms are van Kampen squares. □

For more concise formulations, we define:

Definition 5.3 A *van Kampen setup* in a category $\mathbf{C}$ for a square as in Def. 5.1(i) is a commuting cube in $\mathbf{C}$ as in Def. 5.1(ii) where the bottom square is a pushout and the two back squares are pullbacks. □

For reference, we expand this into the collagory setting:

Lemma 5.4 In a collagory $\mathbf{C}$, a van Kampen setup in $\mathsf{Map}\,\mathbf{C}$ means that the following hold:

Bottom pushout:

$$\begin{array}{lllllllll}
G ; N^{\smile} & = & (M^{\smile} ; F)^{\boxast} & G^{\smile} ; G \sqcup N^{\smile} ; N & = & \mathbb{I}_{\mathcal{D}} & G ; G^{\smile} & = & (M^{\smile} ; F)^{\triangleright} \\
& & & & & & N ; N^{\smile} & = & (M^{\smile} ; F)^{\triangleleft}
\end{array}$$

Back pullbacks:

$$\begin{array}{lllllllll}
c^{\smile} ; m & = & M ; a^{\smile} & c ; c^{\smile} \sqcap m ; m^{\smile} & = & \mathbb{I}_{\mathcal{C}'} & c^{\smile} ; c & = & \mathbb{I}_{\mathcal{C}} \sqcap M ; a^{\smile} ; a ; M^{\smile} \\
& & & & & & m^{\smile} ; m & = & \mathbb{I}_{\mathcal{A}'} \sqcap a ; M^{\smile} ; M ; a^{\smile} \\
c^{\smile} ; f & = & F ; b^{\smile} & c ; c^{\smile} \sqcap f ; f^{\smile} & = & \mathbb{I}_{\mathcal{C}'} & c^{\smile} ; c & = & \mathbb{I}_{\mathcal{C}} \sqcap F ; b^{\smile} ; b ; F^{\smile} \\
& & & & & & f^{\smile} ; f & = & \mathbb{I}_{\mathcal{B}'} \sqcap b ; F^{\smile} ; F ; b^{\smile}
\end{array}$$

Remaining commutative squares:

$$\begin{array}{lllllllll}
m ; g & = & f ; n & g ; d & = & a ; G & n ; d & = & b ; N
\end{array}$$

□

These equations are now used to prove the following:

Lemma 5.5 In the category of maps $\mathsf{Map}\,\mathbf{C}$ over a collagory $\mathbf{C}$, *pushouts along injective maps are stable under pullbacks*, that is, in a van Kampen setup where M is injective, if the front squares are pullbacks, then the top square is a pushout.

PROOF: Besides the assumptions in Lemma 5.4, we also have in particular the following equations for the top sides of the two front pullbacks:

$$\begin{array}{rcl} g^{\smile} ; g & = & \mathbb{I}_{\mathcal{D}'} \sqcap d ; G^{\smile} ; G ; d^{\smile} \\ n^{\smile} ; n & = & \mathbb{I}_{\mathcal{D}'} \sqcap d ; N^{\smile} ; N ; d^{\smile} \end{array}$$

This gives us (without requiring injectivity of M):

$$\begin{array}{rcll} g^{\smile} ; g \sqcup n^{\smile} ; n & = & \mathbb{I}_{\mathcal{D}'} \sqcap (d ; G^{\smile} ; G ; d^{\smile} \sqcup d ; N^{\smile} ; N ; d^{\smile}) & \text{front pullbacks} \\ & = & \mathbb{I}_{\mathcal{D}'} \sqcap d ; (G^{\smile} ; G \sqcup N^{\smile} ; N) ; d^{\smile} & \text{join-distr.} \\ & = & \mathbb{I}_{\mathcal{D}'} \sqcap d ; \mathbb{I}_{\mathcal{D}} ; d^{\smile} & \text{bottom pushout} \\ & = & \mathbb{I}_{\mathcal{D}'} \sqcap d ; d^{\smile} & \text{identity law} \\ & = & \mathbb{I}_{\mathcal{D}'} & d \text{ total} \end{array}$$

With Lemmas 4.4 and 4.11, injectivity of M implies injectivity of m, N, and n. Therefore we also have cross-commutativity $G ; N^{\smile} = M^{\smile} ; F$. With this, we can establish also cross-commutativity for the top pushout:

$$\begin{array}{rcll} g ; n^{\smile} & = & g ; n^{\smile} \sqcap g ; d ; d^{\smile} ; n^{\smile} & d \text{ total} \\ & = & (g \sqcap g ; d ; d^{\smile} ; n^{\smile} ; n) ; n^{\smile} & n \text{ difunctional} \\ & = & (g \sqcap a ; G ; N^{\smile} ; b^{\smile} ; n) ; n^{\smile} & \text{front squares commute} \\ & = & (g \sqcap a ; M^{\smile} ; F ; b^{\smile} ; n) ; n^{\smile} & \text{bottom pushout} \\ & = & (g \sqcap a ; M^{\smile} ; c^{\smile} ; f ; n) ; n^{\smile} & \text{back pullback} \\ & = & (g \sqcap a ; M^{\smile} ; c^{\smile} ; m ; g) ; n^{\smile} & \text{top square commutes} \\ & = & (g \sqcap a ; M^{\smile} ; M ; a^{\smile} ; g) ; n^{\smile} & \text{left pullback} \\ & = & (\mathbb{I} \sqcap a ; M^{\smile} ; M ; a^{\smile}) ; g ; n^{\smile} & \text{allegory property [Kah01, A.2.1(v)]} \\ & = & m^{\smile} ; m ; g ; n^{\smile} & \text{back pullback} \\ & = & m^{\smile} ; f ; n ; n^{\smile} & \text{top square commutativity} \\ & = & m^{\smile} ; f & n \text{ total and injective} \end{array}$$

□

Theorem 5.6 In the category of maps $\mathsf{Map}\,\mathbf{C}$ over a collagory $\mathbf{C}$, *pushouts along injective maps are van Kampen squares.*

PROOF: Since Lemma 5.5 already showed the "only if" part of the definition of can Kampen squares, we only need to show that, in a van Kampen setup where M is injective, if the top square is a pushout, then the front squares are pullbacks.

With Lemmas 4.4 and 4.11, injectivity of M implies injectivity of N, m, and n. With Lemma 4.10 we obtain simpler forms for the pushout equations for the top and bottom squares — we show only those we will use below:

$$
\begin{array}{rclcrclcrcl}
G ; N^{\smile} & = & M^{\smile} ; F & & & & & & G ; G^{\smile} & = & \mathbb{I}_{\mathcal{A}} \sqcup M^{\smile} ; F ; F^{\smile} ; M \\
g ; n^{\smile} & = & m^{\smile} ; f & & g^{\smile} ; g \sqcup n^{\smile} ; n & = & \mathbb{I}_{\mathcal{D}'} & & g ; g^{\smile} & = & \mathbb{I}_{\mathcal{A}'} \sqcup m^{\smile} ; f ; f^{\smile} ; m
\end{array}
$$

This first gives us $d ; N^{\smile} = n^{\smile} ; b$:

$$
\begin{array}{rcll}
d ; N^{\smile} & = & (g^{\smile} ; g \sqcup n^{\smile} ; n) ; d ; N^{\smile} & \text{top pushout} \\
& = & g^{\smile} ; g ; d ; N^{\smile} \sqcup n^{\smile} ; n ; d ; N^{\smile} & \text{distributivity} \\
& = & g^{\smile} ; a ; G ; N^{\smile} \sqcup n^{\smile} ; b ; N ; N^{\smile} & \text{front squares comm.} \\
& = & g^{\smile} ; a ; M^{\smile} ; F \sqcup n^{\smile} ; b & \text{bottom pushout, } N \text{ inj.} \\
& = & g^{\smile} ; m^{\smile} ; c ; F \sqcup n^{\smile} ; b & \text{left pullback} \\
& = & n^{\smile} ; f^{\smile} ; f ; b \sqcup n^{\smile} ; b & \text{top, back squares commute} \\
& = & n^{\smile} ; b & f \text{ univalent}
\end{array}
$$

Similarly, we can also derive $d ; G^{\smile} = g^{\smile} ; a$:

$$
\begin{array}{rcll}
d ; G^{\smile} & = & (g^{\smile} ; g \sqcup n^{\smile} ; n) ; d ; G^{\smile} & \text{top pushout} \\
& = & g^{\smile} ; g ; d ; G^{\smile} \sqcup n^{\smile} ; n ; d ; G^{\smile} & \text{distributivity} \\
& = & g^{\smile} ; a ; G ; G^{\smile} \sqcup n^{\smile} ; b ; N ; G^{\smile} & \text{front squares comm.} \\
& = & g^{\smile} ; a ; (\mathbb{I}_{\mathcal{D}} \sqcup M^{\smile} ; F ; F^{\smile} ; M) \sqcup n^{\smile} ; b ; F^{\smile} ; M & \text{bottom pushout} \\
& = & g^{\smile} ; a \sqcup g^{\smile} ; m^{\smile} ; c ; F ; F^{\smile} ; M \sqcup n^{\smile} ; f^{\smile} ; c ; M & \text{back pullbacks} \\
& = & g^{\smile} ; a \sqcup g^{\smile} ; m^{\smile} ; f ; b ; F^{\smile} ; M \sqcup g^{\smile} ; m^{\smile} ; m ; a & \text{back, top squ. comm.} \\
& = & g^{\smile} ; a \sqcup g^{\smile} ; m^{\smile} ; f ; b ; F^{\smile} ; M & g^{\smile} ; m^{\smile} ; m ; a \sqsubseteq g^{\smile} ; a \\
& = & g^{\smile} ; a \sqcup g^{\smile} ; m^{\smile} ; f ; f^{\smile} ; c ; M & \text{back pullback} \\
& = & g^{\smile} ; a \sqcup g^{\smile} ; m^{\smile} ; f ; f^{\smile} ; m ; a & \text{left square comm.} \\
& = & g^{\smile} ; a & \text{see below}
\end{array}
$$

The last step is justified since:

$$
\begin{array}{rcll}
g^{\smile} ; m^{\smile} ; f ; f^{\smile} ; m ; a & \sqsubseteq & g^{\smile} ; g ; g^{\smile} ; a & m^{\smile} ; f ; f^{\smile} ; m \sqsubseteq g ; g^{\smile} \\
& = & g^{\smile} ; a & g \text{ difunctional}
\end{array}
$$

The second tabulation condition on the “injective side” follows immediately:

$$
\begin{array}{rcll}
n ; n^{\smile} \sqcap b ; b^{\smile} & = & \mathbb{I}_{\mathcal{B}'} \sqcap b ; b^{\smile} & n \text{ total and injective} \\
& = & \mathbb{I}_{\mathcal{B}'} & b \text{ total}
\end{array}
$$

The other side requires more effort:

$$
\begin{array}{rcll}
g ; g^{\smile} \sqcap a ; a^{\smile} & = & (\mathbb{I}_{\mathcal{A}'} \sqcup m^{\smile} ; f ; f^{\smile} ; m) \sqcap a ; a^{\smile} & \text{top pushout} \\
& = & (\mathbb{I}_{\mathcal{A}'} \sqcap a ; a^{\smile}) \sqcup (m^{\smile} ; f ; f^{\smile} ; m \sqcap a ; a^{\smile}) & \text{distributivity} \\
& = & \mathbb{I}_{\mathcal{A}'} \sqcup (m^{\smile} ; f ; f^{\smile} ; m \sqcap a ; a^{\smile}) & a \text{ total} \\
& = & \mathbb{I}_{\mathcal{A}'} & \text{(see below)}
\end{array}
$$

The last step is justified by showing the inclusion $m^{\smile} ; f ; f^{\smile} ; m \sqcap a ; a^{\smile} \sqsubseteq \mathbb{I}_{\mathcal{A}'}$:

$$\begin{array}{rcll} m^{\smile} ; f ; f^{\smile} ; m \sqcap a ; a^{\smile} & \sqsubseteq & m^{\smile} ; (f ; f^{\smile} \sqcap m ; a ; a^{\smile} ; m^{\smile}) ; m & \text{modal rule} \\ & = & m^{\smile} ; (f ; f^{\smile} \sqcap c ; M ; M^{\smile} ; c^{\smile}) ; m & \text{left square commutes} \\ & = & m^{\smile} ; (f ; f^{\smile} \sqcap c ; c^{\smile}) ; m & M \text{ total and injective} \\ & = & m^{\smile} ; \mathbb{I}_{\mathcal{C}'} ; m & \text{back pullback} \\ & \sqsubseteq & \mathbb{I}_{\mathcal{A}'} & m \text{ univalent} \end{array}$$ □

The main result of this section is now an immediate consequence of this theorem; note that we do not need difunctional (or transitive) closure for this:

Corollary 5.7 For a bi-tabular collagory $\mathbf{C}$ where all monos in $\mathsf{Map}\,\mathbf{C}$ are injective in $\mathbf{C}$, the mapping category $\mathsf{Map}\,\mathbf{C}$ is adhesive. □

(The restriction on monic mappings is necessary since there might, for example, be an object $\mathcal{A}$ in $\mathbf{C}$ for which the only mapping with target $\mathcal{A}$ is $\mathbb{I}_{\mathcal{A}}$; in that case, all mappings $f : \mathcal{A} \to \mathcal{B}$ would automatically be monos in $\mathsf{Map}\,\mathbf{C}$ regardless whether they are injective in $\mathbf{C}$. Note that f (together with identities) itself forms a tabulation and a co-tabulation for f.)

This result immediately makes the rewriting concepts and results from [LS04], including the local Church-Rosser theorem and the concurrency theorem, available for DPO rewriting defined via tabulations and co-tabulations in the context of collagories.

6 Collagories of Semi-Unary Algebras and Bisimulations

In [Kah01, Kah04], relational homomorphisms between unary algebras have been shown to form a distributive allegory. In this section we generalise this result to collagories by allowing constant symbols and in turn dropping the zero law requirement.

Most of the mathematical content of this section has been presented and proven in more detail in [Kah01, Kah04]. Besides the proof of Theorem 6.6, also the reformulation using the sort-indexed product category and the forgetful functor $\mathcal{U}_\Sigma$ is new.

Definition 6.1 A *signature* is a tuple $(\mathcal{S}, \mathcal{F}, \mathsf{src}, \mathsf{trg})$ consisting of

- a set $\mathcal{S}$ of *sorts*,
- a set $\mathcal{F}$ of *function symbols*,
- a mapping $\mathsf{src} : \mathcal{F} \to \mathcal{S}^*$ associating with every function symbol the list of its *source sorts*, and
- a mapping $\mathsf{trg} : \mathcal{F} \to \mathcal{S}$ associating with every function symbol its *target sort*.

Such a signature is called *semi-unary* if $\mathsf{length}(\mathsf{src}(f)) \leq 1$ for each $f : \mathcal{F}$, and *unary* if $\mathsf{length}(\mathsf{src}(f)) = 1$ for each $f : \mathcal{F}$. □

For a function symbol $f : \mathcal{F}$, we usually employ the shorthand "$f : s_1 \times \cdots \times s_n \to t$" instead of the rather verbose "$\mathsf{src}(f) = \langle s_1, \ldots, s_n \rangle$ and $\mathsf{trg}(f) = t$". For a zero-ary function symbol, also called *constant symbol*, we write "$f : \mathbb{1} \to t$".

The following example signatures will be used for discussion and results in sections 7 and 8:

$$\mathsf{sigGraph} := \langle\ \textbf{sorts: } \mathsf{V}, \mathsf{E} \quad \textbf{ops: } \mathsf{s}, \mathsf{t} : \mathsf{E} \to \mathsf{V}\ \rangle$$

$$\mathsf{sigPointedSet} := \langle\ \textbf{sorts: } \mathsf{S} \quad \textbf{ops: } \mathsf{point} : \mathbb{1} \to \mathsf{S}\ \rangle$$

$$\mathsf{sigPoint} := \langle\ \textbf{sorts: } \mathsf{P} \quad \textbf{ops: }\ \rangle$$

$$\mathsf{sigPointed} := \langle\ \textbf{sorts: } \mathsf{P}, \mathsf{O} \quad \textbf{ops: } \mathsf{p} : \mathsf{P} \to \mathsf{O}\ \rangle$$

$$\mathsf{sigType} := \langle\ \textbf{sorts: } \mathsf{T} \quad \textbf{ops: }\ \rangle$$

$$\mathsf{sigTyped} := \langle\ \textbf{sorts: } \mathsf{O}, \mathsf{T} \quad \textbf{ops: } \mathsf{t} : \mathsf{O} \to \mathsf{T}\ \rangle$$

$$\mathsf{sigNELabels} := \langle\ \textbf{sorts: } \mathsf{NL}, \mathsf{EL} \quad \textbf{ops: }\ \rangle$$

$$\mathsf{sigLGraph} := \langle\ \textbf{sorts: } \mathsf{N}, \mathsf{E}, \mathsf{NL}, \mathsf{EL} \quad \textbf{ops: } \mathsf{s}, \mathsf{t} : \mathsf{E} \to \mathsf{N},\ \mathsf{n} : \mathsf{N} \to \mathsf{NL},\ \mathsf{e} : \mathsf{E} \to \mathsf{EL}\ \rangle$$

Definition 6.2 For a set $\mathcal{S}$ (of *sorts*) and a category $\mathbf{C}$, we define $\mathbf{C}^{\mathcal{S}}$, the *$\mathcal{S}$-indexed product category of* $\mathbf{C}$, as follows:

- an object $\mathcal{A}$ of $\mathbf{C}^{\mathcal{S}}$ consists of $\mathbf{C}$-objects $s^{\mathcal{A}}$ for every $s : \mathcal{S}$;
- a morphism $\Phi : \mathcal{A} \to \mathcal{B}$ of $\mathbf{C}^{\mathcal{S}}$ is an $\mathcal{S}$-indexed family of $\mathbf{C}$-morphisms $\Phi = (\Phi_s)_{s:\mathcal{S}}$ such that $\Phi_s : s^{\mathcal{A}} \to s^{\mathcal{B}}$ for every sort $s : \mathcal{S}$.
- composition $;^{\mathcal{S}}$ and identities $\mathbb{I}^{\mathcal{S}}$ are defined component-wise;
- if $\mathbf{C}$ is an allegory, then inclusion $\sqsubseteq^{\mathcal{S}}$, meet $\sqcap^{\mathcal{S}}$ and converse are defined component-wise;
- if $\mathbf{C}$ is collagory, then join $\sqcup^{\mathcal{S}}$ is defined component-wise. □

One easily verifies that the resulting $\mathcal{S}$-indexed product categories, allegories, and collagories all satisfy the respective axioms.

When defining Σ-algebras in the presence of binary function symbols, we need several technical conditions on direct products [Kah01, Def. 3.1.12]; for the current study, we can do without direct products (at the cost of some duplication of formalisation for unary and zero-ary function symbols), but we still need allegories for the characterisation of mappings:

Definition 6.3 Given a signature $\Sigma = (\mathcal{S}, \mathcal{F}, \mathsf{src}, \mathsf{trg})$ and an allegory $\mathbf{C}$, which has to have a unit $\mathbb{1}$ if Σ contains constant symbols, an *abstract Σ-algebra* over $\mathbf{C}$ consists of the following items:

- an object $\mathcal{A}$ of $\mathbf{C}^{\mathcal{S}}$,
- for every function symbol $f{:}\mathcal{F}$ with $f : s \to t$ a mapping $f^{\mathcal{A}} : s^{\mathcal{A}} \to t^{\mathcal{A}}$ in $\mathbf{C}$.
- for every constant symbol $c{:}\mathcal{F}$ with $c : \mathbb{1} \to t$ a mapping $c^{\mathcal{A}} : \mathbb{1} \to t^{\mathcal{A}}$ in $\mathbf{C}$. □

It is important to note that, where we use sets as carriers, we have no restriction to non-empty sets — unlike most of the universal algebra literature.

Since we use this definition to construct an allegory with abstract Σ-algebras as objects, the generality of discussing *abstract* Σ-algebras over allegories allows us to stack this construction at no cost at all, with possibly different signatures at every level, building for example graphs where the nodes and edges are hypergraphs and hypergraph morphisms.

The morphisms in allegories of Σ-algebras have to behave "essentially like relations", and so it is only natural that we consider a relational generalisation of conventional (functional) Σ-homomorphisms. For arbitrary signatures, this has been presented in [Kah01]. For unary signatures, one naturally starts with defining L-simulations satisfying $\Phi_s^{\smile} ; f^{\mathcal{A}} \sqsubseteq f^{\mathcal{B}} ; \Phi_t^{\smile}$ according to de Roever and Engelhardt [dRE98], and then proceeds to L-simulations for which their converse is an L-simulation, too; these are called "bisimulations" in [Kah04].

Definition 6.4 Let a signature $\Sigma = (\mathcal{S}, \mathcal{F}, \mathsf{src}, \mathsf{trg})$, an allegory $\mathbf{C}$, and two abstract Σ-algebras $\mathcal{A}$ and $\mathcal{B}$ over $\mathbf{C}$ be given.

A *Σ-bisimulation from $\mathcal{A}$ to $\mathcal{B}$* is a $\mathbf{C}^{\mathcal{S}}$-morphisms from $\mathcal{A}$ to $\mathcal{B}$ such that for every function symbol $f \in \mathcal{F}$ with $f : s \to t$ and every constant symbol $c \in \mathcal{F}$ with $c : \mathbb{1} \to t$ the following inclusions hold:

$$\Phi_s ; f^{\mathcal{B}} \sqsubseteq f^{\mathcal{A}} ; \Phi_t, \qquad \text{and} \qquad c^{\mathcal{B}} \sqsubseteq c^{\mathcal{A}} ; \Phi_t \ . \qquad \square$$

In the allegory $\mathbf{C}$, this gives rise to the following sub-commuting diagrams (including one for the n-ary case):

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "c^{\mathcal{A}}"] \arrow[d, "\mathbb{I}_{\mathbb{1}}"'] \arrow[dr, phantom, "\sqsubseteq"] & t^{\mathcal{A}} \arrow[d, "\Phi_t"] \\
\mathbb{1} \arrow[r, "c^{\mathcal{B}}"'] & t^{\mathcal{B}}
\end{tikzcd}
\qquad
\begin{tikzcd}
s^{\mathcal{A}} \arrow[r, "f^{\mathcal{A}}"] \arrow[d, "\Phi_s"'] \arrow[dr, phantom, "\sqsubseteq"] & t^{\mathcal{A}} \arrow[d, "\Phi_t"] \\
s^{\mathcal{B}} \arrow[r, "f^{\mathcal{B}}"'] & t^{\mathcal{B}}
\end{tikzcd}
\qquad
\begin{tikzcd}
s_1^{\mathcal{A}} \times \cdots \times s_n^{\mathcal{A}} \arrow[r, "g^{\mathcal{A}}"] \arrow[d, "\Phi_{s_1} \times \cdots \times \Phi_{s_n}"'] \arrow[dr, phantom, "\sqsubseteq"] & t^{\mathcal{A}} \arrow[d, "\Phi_t"] \\
s_1^{\mathcal{B}} \times \cdots \times s_n^{\mathcal{B}} \arrow[r, "g^{\mathcal{B}}"'] & t^{\mathcal{B}}
\end{tikzcd}$$

Using Σ-algebras over $\mathbf{C}$ as objects and Σ-bisimulations as morphisms defines a category $\mathbf{C}^{\Sigma}$ with an obvious "underlying" functor $\mathcal{U}_{\Sigma} : \mathbf{C}^{\Sigma} \to \mathbf{C}^{\mathcal{S}}$.

This "forgetful" functor $\mathcal{U}_{\Sigma}$ is faithful. If $\mathbf{C}$ is an allegory, then $\mathcal{U}_{\Sigma}$ reflects inclusion, meets and converse in the sense that these can be defined for $\mathbf{C}^{\Sigma}$ via their $\mathcal{U}_{\Sigma}$ images. Therefore, $\mathbf{C}^{\Sigma}$ is an allegory, too [Kah01, Thm. 3.2.6].

We may observe a few simple facts:

- If $\mathbf{C}$ contains an initial object $\varnothing$, and Σ contains no constants, then we obtain an initial object $\mathbb{0}_{\Sigma}$ in $\mathbf{C}^{\Sigma}$ by choosing $s^{\mathbb{0}_{\Sigma}} = \varnothing$ for each sort s and $f^{\mathbb{0}_{\Sigma}} = \mathbb{I}_{\varnothing}$ for each function symbol f.
- If $\mathbf{C}$ contains a unit $\mathbb{1}$, then we obtain a unit $\mathbb{1}_{\Sigma}$ in $\mathbf{C}^{\Sigma}$ by choosing $s^{\mathbb{1}_{\Sigma}} = \mathbb{1}$ for each sort s and $f^{\mathbb{1}_{\Sigma}} = \mathbb{I}_{\mathbb{1}}$ for each function symbol f.

Conventional Σ-algebra homomorphisms are just mappings in the allegory Rel^{Σ} of concrete Σ-algebras over the allegory Rel of sets and concrete relations.

If Σ contains a constant symbol, then even if the allegory $\mathbf{C}$ has least morphisms, then least homomorphisms in $\mathbf{C}^{\mathcal{S}}$ are not generally in the range of $\mathcal{U}_\Sigma$, and even if $\mathbf{C}^\Sigma$ does have least morphisms, the zero law will in general not hold for them, no matter whether it holds in $\mathbf{C}$.

If Σ contains a function symbol of arity at least 2, then even if $\mathbf{C}$ is an upper-semilattice category, then $\mathcal{U}_\Sigma$ does not reflect joins, in the sense that $\mathcal{U}_\Sigma(\Phi) \sqcup^{\mathcal{S}} \mathcal{U}_\Sigma(\Psi)$ is not necessarily in the range of $\mathcal{U}_\Sigma$. Furthermore, even if $\mathbf{C}^\Sigma$ has joins, composition will, in presence of function symbols of arity at least 2, in general not distribute over these joins (since non-empty joins do not distribute over the product $\times$ occurring in the homomorphism condition) so $\mathbf{C}^\Sigma$ will not be an upper-semilattice category.

For semi-unary signatures, however, $\mathcal{U}_\Sigma$ does reflect joins:

Lemma 6.5 If $\mathbf{C}$ is an upper-semilattice category, Σ is a semi-unary signature, and $\Phi, \Psi : \mathcal{A} \to \mathcal{B}$ are two Σ-bisimulations, then $\Phi \sqcup^{\mathcal{S}} \Psi$ is a Σ-bisimulation, too, and is the join in $\mathbf{C}^\Sigma$ of Φ and Ψ, that is, $\Phi \sqcup^\Sigma \Psi = \Phi \sqcup^{\mathcal{S}} \Psi$.

Proof: We need to check the bisimulation conditions for unary function symbols $f : s \to t$ and for constant symbols $c : \mathbb{1} \to t$:

$$\begin{array}{rcl}
(\Phi \sqcup^{\mathcal{S}} \Psi)_s \mathbin{;} f^{\mathcal{B}} & = & (\Phi_s \sqcup \Psi_s) \mathbin{;} f^{\mathcal{B}} \;=\; \Phi_s \mathbin{;} f^{\mathcal{B}} \sqcup \Psi_s \mathbin{;} f^{\mathcal{B}} \\
& \sqsubseteq & f^{\mathcal{A}} \mathbin{;} \Phi_t \sqcup f^{\mathcal{A}} \mathbin{;} \Psi_t \;=\; f^{\mathcal{A}} \mathbin{;} (\Phi_t \sqcup \Psi_t) \;=\; f^{\mathcal{A}} \mathbin{;} (\Phi \sqcup^{\mathcal{S}} \Psi)_t \\
c^{\mathcal{B}} & \sqsubseteq & c^{\mathcal{A}} \mathbin{;} \Phi_t \sqcup c^{\mathcal{A}} \mathbin{;} \Psi_t \;=\; c^{\mathcal{A}} \mathbin{;} (\Phi_t \sqcup \Psi_t) \;=\; c^{\mathcal{A}} \mathbin{;} (\Phi \sqcup^{\mathcal{S}} \Psi)_t
\end{array}$$

The equation $\Phi \sqcup^\Sigma \Psi = \Phi \sqcup^{\mathcal{S}} \Psi$ follows from the reflection of inclusion by $\mathcal{U}_\Sigma$. □

Given the closure of Σ-bisimulations under the converse, meet, and join operations in $\mathbf{C}^{\mathcal{S}}$, properties of $\mathbf{C}$-morphisms for these operations are inherited by Σ-bisimulations because of the component-wise definitions, and we obtain:

Theorem 6.6 If Σ is a semi-unary signature and $\mathbf{C}$ is a collagory, then $\mathbf{C}^\Sigma$ is a collagory, too. □

If $\mathbf{C}$ has tabulations (respectively co-tabulations), the sort-indexed product category $\mathbf{C}^{\mathcal{S}}$ obviously has tabulations (respectively co-tabulations), too, and they can be calculated component-wise. Perhaps surprisingly, these can be extended to the collagory $\mathbf{C}^\Sigma$ of bisimulations between Σ-algebras without problems; we just need to provide definitions for the function symbols of the "new" objects, and verify all relevant conditions:

Theorem 6.7 If $\Sigma = (\mathcal{S}, \mathcal{F}, \mathsf{src}, \mathsf{trg})$ is a semi-unary signature and $\mathbf{C}$ is an allegory, and $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation in $\mathbf{C}^{\mathcal{S}}$ of the Σ-bisimulation $V : \mathcal{B} \to \mathcal{C}$, i.e., for each sort $s : \mathcal{S}$, $\mathcal{B} \xleftarrow{P_s} \mathcal{A} \xrightarrow{Q_s} \mathcal{C}$ is a tabulation of $V_s : s^{\mathcal{B}} \to s^{\mathcal{C}}$, then we define for each function symbol $f : s \to t$ and each constant symbol $c : \mathbb{1} \to t$ in Σ:

$$\begin{array}{rcl}
f^{\mathcal{A}} & := & P_s \mathbin{;} f^{\mathcal{B}} \mathbin{;} P_t^{\smile} \sqcap Q_s \mathbin{;} f^{\mathcal{C}} \mathbin{;} Q_t^{\smile} \\
c^{\mathcal{A}} & := & c^{\mathcal{B}} \mathbin{;} P_t^{\smile} \sqcap c^{\mathcal{C}} \mathbin{;} Q_t^{\smile}
\end{array}$$

Then $\mathcal{A}$ turns into a Σ-algebra and P and Q are Σ-bisimulations, too, so $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation in $\mathbf{C}^\Sigma$.

Proof: We first show the bisimulation conditions for P; those for Q follow analogously:

$$
\begin{array}{rcll}
P_s ; f^{\mathcal{B}} & = & (P_s \sqcap Q_s ; Q_s^{\smile} ; P_s) ; f^{\mathcal{B}} & Q_s \text{ total} \\
& = & (P_s \sqcap Q_s ; V_s^{\smile}) ; f^{\mathcal{B}} & \text{tabulation of } V_s \\
& \sqsubseteq & P_s ; f^{\mathcal{B}} \sqcap Q_s ; V_s^{\smile} ; f^{\mathcal{B}} & \text{meet-subdistributivity} \\
& \sqsubseteq & P_s ; f^{\mathcal{B}} \sqcap Q_s ; f^{\mathcal{C}} ; V_t^{\smile} & V^{\smile} \text{ bisimulation} \\
& = & P_s ; f^{\mathcal{B}} \sqcap Q_s ; f^{\mathcal{C}} ; Q_t^{\smile} ; P_t & \text{tabulation of } V_t \\
& = & (P_s ; f^{\mathcal{B}} ; P_t^{\smile} \sqcap Q_s ; f^{\mathcal{C}} ; Q_t^{\smile}) ; P_t & P_t \text{ univalent} \\
& = & f^{\mathcal{A}} ; P_t & \text{Def. } f^{\mathcal{A}} \\
c^{\mathcal{B}} & = & c^{\mathcal{B}} \sqcap c^{\mathcal{C}} ; V_t^{\smile} & V^{\smile} \text{ bisimulation} \\
& = & c^{\mathcal{B}} \sqcap c^{\mathcal{C}} ; Q_t^{\smile} ; P_t & \text{tabulation of } V_t \\
& = & (c^{\mathcal{B}} ; P_t^{\smile} \sqcap c^{\mathcal{C}} ; Q_t^{\smile}) ; P_t & P_t \text{ univalent} \\
& = & c^{\mathcal{A}} ; P_t & \text{Def. } c^{\mathcal{A}}
\end{array}
$$

Next we show that $f^{\mathcal{A}}$ and $c^{\mathcal{A}}$ are univalent:

$$
\begin{array}{rcll}
(f^{\mathcal{A}})^{\smile} ; f^{\mathcal{A}} & = & (P_t ; (f^{\mathcal{B}})^{\smile} ; P_s^{\smile} \sqcap Q_t ; (f^{\mathcal{C}})^{\smile} ; Q_s^{\smile}) & \\
& & ; (P_s ; f^{\mathcal{B}} ; P_t^{\smile} \sqcap Q_s ; f^{\mathcal{C}} ; Q_t^{\smile}) & \\
& \sqsubseteq & P_t ; (f^{\mathcal{B}})^{\smile} ; P_s^{\smile} ; P_s ; f^{\mathcal{B}} ; P_t^{\smile} \sqcap & \\
& & Q_t ; (f^{\mathcal{C}})^{\smile} ; Q_s^{\smile} ; Q_s ; f^{\mathcal{C}} ; Q_t^{\smile} & \\
& \sqsubseteq & P_t ; (f^{\mathcal{B}})^{\smile} ; f^{\mathcal{B}} ; P_t^{\smile} \sqcap Q_t ; (f^{\mathcal{C}})^{\smile} ; f^{\mathcal{C}} ; Q_t^{\smile} & P_s,\ Q_s \text{ univalent} \\
& \sqsubseteq & P_t ; P_t^{\smile} \sqcap Q_t ; Q_t^{\smile} & f^{\mathcal{B}},\ f^{\mathcal{C}} \text{ univalent} \\
& = & \mathbb{I}_{t^{\mathcal{A}}} & \text{tabulation of } V_t \\
(c^{\mathcal{A}})^{\smile} ; c^{\mathcal{A}} & = & (P_t ; (c^{\mathcal{B}})^{\smile} \sqcap Q_t ; (c^{\mathcal{C}})^{\smile}) ; (c^{\mathcal{B}} ; P_t^{\smile} \sqcap c^{\mathcal{C}} ; Q_t^{\smile}) & \\
& \sqsubseteq & P_t ; (c^{\mathcal{B}})^{\smile} ; c^{\mathcal{B}} ; P_t^{\smile} \sqcap Q_t ; (c^{\mathcal{C}})^{\smile} ; c^{\mathcal{C}} ; Q_t^{\smile}) & \\
& \sqsubseteq & P_t ; P_t^{\smile} \sqcap Q_t ; Q_t^{\smile} & c^{\mathcal{B}},\ c^{\mathcal{C}} \text{ univalent} \\
& = & \mathbb{I}_{t^{\mathcal{A}}} & \text{tabulation of } V_t
\end{array}
$$

For showing totality of $f^{\mathcal{A}}$ and $c^{\mathcal{A}}$, we use all the above:

$$
\begin{array}{rcll}
f^{\mathcal{A}} ; (f^{\mathcal{A}})^{\smile} & = & f^{\mathcal{A}} ; (P_t ; (f^{\mathcal{B}})^{\smile} ; P_s^{\smile} \sqcap Q_t ; (f^{\mathcal{C}})^{\smile} ; Q_s^{\smile}) & \\
& = & f^{\mathcal{A}} ; P_t ; (f^{\mathcal{B}})^{\smile} ; P_s^{\smile} \sqcap f^{\mathcal{A}} ; Q_t ; (f^{\mathcal{C}})^{\smile} ; Q_s^{\smile} & f^{\mathcal{A}} \text{ univalent} \\
& \sqsupseteq & P_s ; f^{\mathcal{B}} ; (f^{\mathcal{B}})^{\smile} ; P_s^{\smile} \sqcap Q_s ; f^{\mathcal{C}} ; (f^{\mathcal{C}})^{\smile} ; Q_s^{\smile} & P,\ Q \text{ bisim.} \\
& \sqsupseteq & P_s ; P_s^{\smile} \sqcap Q_s ; Q_s^{\smile} & f^{\mathcal{B}},\ f^{\mathcal{C}} \text{ total} \\
& = & \mathbb{I}_{s^{\mathcal{A}}} & \text{tabulation of } V_s \\
c^{\mathcal{A}} ; (c^{\mathcal{A}})^{\smile} & = & c^{\mathcal{A}} ; (P_t ; (c^{\mathcal{B}})^{\smile} \sqcap Q_t ; (c^{\mathcal{C}})^{\smile}) & \\
& = & c^{\mathcal{A}} ; P_t ; (c^{\mathcal{B}})^{\smile} \sqcap c^{\mathcal{A}} ; Q_t ; (c^{\mathcal{C}})^{\smile} & c^{\mathcal{A}} \text{ univalent} \\
& = & c^{\mathcal{B}} ; (c^{\mathcal{B}})^{\smile} \sqcap c^{\mathcal{C}} ; (c^{\mathcal{C}})^{\smile} & P,\ Q \text{ bisim.} \\
& \sqsupseteq & \mathbb{I}_{\mathbb{1}} & c^{\mathcal{B}},\ c^{\mathcal{C}} \text{ total} \qquad \square
\end{array}
$$

Theorem 6.8 If $\Sigma = (\mathcal{S}, \mathcal{F}, \mathsf{src}, \mathsf{trg})$ is a semi-unary signature and $\mathbf{C}$ is a collagory, and $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation in $\mathbf{C}^{\mathcal{S}}$ of the Σ-bisimulation $W : \mathcal{B} \to \mathcal{C}$, i.e., for each sort $s : \mathcal{S}$, $\mathcal{B} \xrightarrow{R_s} \mathcal{D} \xleftarrow{S_s} \mathcal{C}$ is a tabulation of $W_s : s^{\mathcal{B}} \to s^{\mathcal{C}}$, then we define for each function symbol $f : s \to t$ and each constant symbol $c : \mathbb{1} \to t$ in Σ:

$$\begin{aligned} f^{\mathcal{D}} &:= R_s^{\smile} ; f^{\mathcal{B}} ; R_t \sqcup S_s^{\smile} ; f^{\mathcal{C}} ; S_t \\ c^{\mathcal{D}} &:= c^{\mathcal{B}} ; R_t \sqcup c^{\mathcal{C}} ; S_t \end{aligned}$$

Then $\mathcal{D}$ turns into a Σ-algebra and R and S are Σ-bisimulations, too, so $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation in $\mathbf{C}^{\Sigma}$.

Proof: We first show the bisimulation conditions for R; those for S follow analogously:

$$\begin{array}{rcll} R_s ; f^{\mathcal{D}} & = & R_s ; (R_s^{\smile} ; f^{\mathcal{B}} ; R_t \sqcup S_s^{\smile} ; f^{\mathcal{C}} ; S_t) & \text{Def. } f^{\mathcal{A}} \\ & = & R_s ; R_s^{\smile} ; f^{\mathcal{B}} ; R_t \sqcup R_s ; S_s^{\smile} ; f^{\mathcal{C}} ; S_t & \text{join distr.} \\ & = & (\mathbb{I}_{s^{\mathcal{B}}} \sqcup W_s ; W_s^{\smile}) ; f^{\mathcal{B}} ; R_t \sqcup W_s ; f^{\mathcal{C}} ; S_t & \text{co-tabulation of } W_s \\ & = & f^{\mathcal{B}} ; R_t \sqcup W_s ; W_s^{\smile}) ; f^{\mathcal{B}} ; R_t \sqcup W_s ; f^{\mathcal{C}} ; S_t & \text{join distr.} \\ & \sqsubseteq & f^{\mathcal{B}} ; R_t \sqcup W_s ; f^{\mathcal{C}} ; W_t^{\smile} ; R_t \sqcup W_s ; f^{\mathcal{C}} ; S_t & W^{\smile} \text{ bisimulation} \\ & \sqsubseteq & f^{\mathcal{B}} ; R_t \sqcup W_s ; f^{\mathcal{C}} ; S_t & \text{co-tabulation of } W_t \\ & \sqsubseteq & f^{\mathcal{B}} ; R_t \sqcup f^{\mathcal{B}} ; W_t ; S_t & W \text{ bisimulation} \\ & \sqsubseteq & f^{\mathcal{B}} ; R_t & \text{co-tabulation of } W_t \\ c^{\mathcal{D}} & = & c^{\mathcal{B}} ; R_t \sqcup c^{\mathcal{C}} ; S_t & \text{Def. } c^{\mathcal{D}} \\ & = & c^{\mathcal{B}} ; R_t \sqcup c^{\mathcal{B}} ; W_t^{\smile} ; S_t & W^{\smile} \text{ bisimulation} \\ & = & c^{\mathcal{B}} ; R_t & \text{co-tabulation of } W_t \end{array}$$

Totality and univalence of $c^{\mathcal{D}}$ follows immediately from $c^{\mathcal{D}} = c^{\mathcal{B}} ; R_t$ shown above; for $f^{\mathcal{D}}$, we easily obtain totality:

$$\begin{array}{rcll} f^{\mathcal{D}} ; (f^{\mathcal{D}})^{\smile} & = & (R_s^{\smile} ; f^{\mathcal{B}} ; R_t \sqcup S_s^{\smile} ; f^{\mathcal{C}} ; S_t) & \\ & & ; (R_t^{\smile} ; (f^{\mathcal{B}})^{\smile} ; R_s \sqcup S_t^{\smile} ; (f^{\mathcal{C}})^{\smile} ; S_s) & \\ & \sqsupseteq & R_s^{\smile} ; f^{\mathcal{B}} ; R_t ; R_t^{\smile} ; (f^{\mathcal{B}})^{\smile} ; R_s & \\ & & ; S_s^{\smile} ; f^{\mathcal{C}} ; S_t ; S_t^{\smile} ; (f^{\mathcal{C}})^{\smile} ; S_s & \\ & \sqsupseteq & R_s^{\smile} ; f^{\mathcal{B}} ; (f^{\mathcal{B}})^{\smile} ; R_s ; S_s^{\smile} ; f^{\mathcal{C}} ; (f^{\mathcal{C}})^{\smile} ; S_s & R_t,\ S_t \text{ total} \\ & \sqsupseteq & R_s^{\smile} ; R_s ; S_s^{\smile} ; S_s & f^{\mathcal{B}},\ f^{\mathcal{C}} \text{ total} \\ & = & \mathbb{I}_{s^{\mathcal{D}}} & \text{co-tabulation of } W_s \end{array}$$

Univalence of $f^{\mathcal{D}}$:

$$\begin{array}{rcll} (f^{\mathcal{D}})^{\smile} ; f^{\mathcal{D}} & = & (R_t^{\smile} ; (f^{\mathcal{B}})^{\smile} ; R_s \sqcup S_t^{\smile} ; (f^{\mathcal{C}})^{\smile} ; S_s) ; f^{\mathcal{D}} & \\ & = & R_t^{\smile} ; (f^{\mathcal{B}})^{\smile} ; R_s ; f^{\mathcal{D}} \sqcup S_t^{\smile} ; (f^{\mathcal{C}})^{\smile} ; S_s ; f^{\mathcal{D}} & \text{join distr.} \\ & \sqsubseteq & R_t^{\smile} ; (f^{\mathcal{B}})^{\smile} ; f^{\mathcal{B}} ; R_t \sqcup S_t^{\smile} ; (f^{\mathcal{C}})^{\smile} ; f^{\mathcal{C}} ; S_t & R,\ S \text{ bisimul.} \\ & \sqsubseteq & R_t^{\smile} ; R_t \sqcup S_t^{\smile} ; ; S_t & f^{\mathcal{B}},\ f^{\mathcal{C}} \text{ univalent} \\ & = & \mathbb{I}_{t^{\mathcal{D}}} & \text{co-tabulation of } W_t \end{array}$$

□

7 Reducts Along Signature Homomorphisms

While the concept of Σ-algebra is sufficient to capture, for example, unlabelled graphs as $\mathsf{sigGraph}$-algebras, categories of labelled graphs are frequently considered as having *fixed* label sets, which means that only certain sub-categories of $Set^{\mathsf{sigLGraph}}$ are considered.

We use the concept of *reducts* to formalise this in a general way. In the example, we consider the reduct of $Set^{\mathsf{sigLGraph}}$ to the sub-signature $\mathsf{sigNELabels}$. The fixed label sets under consideration form a one-object sub-category $\mathbf{K}$ of $Set^{\mathsf{sigNELabels}}$, and in order to obtain graphs labelled over these label sets, we restrict attention to objects in $Set^{\mathsf{sigLGraph}}$ for which the reduct lies in that sub-category $\mathbf{K}$.

The current section introduces and studies the reduct relator. This is employed in Sect. 8 to implement the restriction of Σ-algebra collagories via reduct-side sub-categories. This single construction principle for generating concrete bi-tabular collagories corresponds, as shown in Corollary 8.7, to several categorical constructions that are known for adhesive categories.

Definition 7.1 Let $\Sigma = (\mathcal{S}, \mathcal{F}, \mathsf{src}, \mathsf{trg})$ and $\Sigma_{\mathrm{R}} = (\mathcal{S}_{\mathrm{R}}, \mathcal{F}_{\mathrm{R}}, \mathsf{src}_{\mathrm{R}}, \mathsf{trg}_{\mathrm{R}})$ be two signatures, and let $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$ be a signature homomorphism.

For any Σ-algebra $\mathcal{A}$, such a signature homomorphism $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$ induces a Σ_{R}-algebra $\mathcal{A}\lfloor\sigma$, the *σ-reduct of* $\mathcal{A}$, in the following way:

- For every sort $r : \mathcal{S}_{\mathrm{R}}$, its carrier is $r^{\mathcal{A}\lfloor\sigma} = (\sigma\ r)^{\mathcal{A}}$;
- for every function symbol $f \in \mathcal{F}_{\mathrm{R}}$, its interpretation is $f^{\mathcal{A}\lfloor\sigma} = (\sigma\ f)^{\mathcal{A}}$. □

It is easy to verify that $\mathcal{A}\lfloor\sigma$ is indeed a Σ_{R}-algebra.

If $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$ is a sub-signature embedding, then we also call $\mathcal{A}\lfloor\sigma$ the Σ_{R}-reduct of $\mathcal{A}$ and write also $\mathcal{A}\lfloor\Sigma_{\mathrm{R}}$.

Since our signatures are a special case of sketches [BW99, Chapters 4,7,8,10], $\lfloor\sigma$ is a special case of what Barr and Wells call "model category functor". We complete the definition and show that is is a relator:

Definition 7.2 For a signature homomorphism $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$, the σ-reduct of a $\mathbf{C}^{\mathcal{S}}$-morphism $\Phi = (\Phi_s)_{s:\mathcal{S}}$ is the $\mathbf{C}^{\mathcal{S}_{\mathrm{R}}}$-morphism $\Phi\lfloor\sigma = ((\Phi\lfloor\sigma)_r)_{r:\mathcal{S}_{\mathrm{R}}}$ with $(\Phi\lfloor\sigma)_r := \Phi_{\sigma\ r}$ for every $r : \mathcal{S}_{\mathrm{R}}$. □

Proposition 7.3 For a signature homomorphism $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$, the σ-reduct of a Σ-bisimulation is a Σ_{R}-bisimulation.

Furthermore, the reduct operation $\lfloor\sigma$ is an allegory relator from $\mathbf{C}^{\Sigma}$ to $\mathbf{C}^{\Sigma_{\mathrm{R}}}$ and therefore also a functor from $\mathsf{Map}\,(\mathbf{C}^{\Sigma})$ to $\mathsf{Map}\,(\mathbf{C}^{\Sigma_{\mathrm{R}}})$.

Proof: Bisimulation property: For any n-ary function symbol (we do not need the restriction to semi-unary signatures here) $f : r_1 \times \cdots \times r_n \to q$ in $\mathcal{F}_{\mathrm{R}}$:

$$\begin{array}{cl} & ((\Phi\lfloor\sigma)_{r_1} \times \cdots \times (\Phi\lfloor\sigma)_{r_n})\,;\, f^{\mathcal{B}\lfloor\sigma} \\ = & (\Phi_{\sigma\, r_1} \times \cdots \times \Phi_{\sigma\, r_n})\,;\,(\sigma\, f)^{\mathcal{B}} \sqsubseteq (\sigma\, f)^{\mathcal{A}}\,;\,\Phi_{\sigma\, q} \ = \ f^{\mathcal{A}\lfloor\sigma}\,;\,(\Phi\lfloor\sigma)_q \end{array}$$

Preservation of identities:

$$\mathbb{I}_{\mathcal{A}}\lfloor\sigma = ((\mathbb{I}_{s^{\mathcal{A}}})_{s\in\mathcal{S}})\lfloor\sigma = ((\mathbb{I}_{(\sigma\, r)^{\mathcal{A}}})_{r\in\mathcal{S}_{\mathrm{R}}} = ((\mathbb{I}_{r^{\mathcal{A}\lfloor\sigma}})_{r\in\mathcal{S}_{\mathrm{R}}} = \mathbb{I}_{\mathcal{A}\lfloor\sigma}$$

Preservation of composition:

$$\begin{array}{rcl}(\Phi \mathbin{;^\Sigma} \Psi)\lfloor\sigma & = & ((\Phi \mathbin{;} \Psi)_{\sigma\, r})_{r\in\mathcal{S}_\mathrm{R}} \;=\; (\Phi_{\sigma\, r} \mathbin{;} \Psi_{\sigma\, r})_{r\in\mathcal{S}_\mathrm{R}} \\ & = & (\Phi_{\sigma\, r})_{r\in\mathcal{S}_\mathrm{R}} \mathbin{;} (\Psi_{\sigma\, r})_{r\in\mathcal{S}_\mathrm{R}} \;=\; (\Phi\lfloor\sigma) \mathbin{;^{\Sigma_\mathrm{R}}} (\Psi\lfloor\sigma)\end{array}$$

Preservation of converse:

$$\Phi^{\smile}\lfloor\sigma = ((\Phi^{\smile})_{\sigma\, r})_{r\in\mathcal{S}_\mathrm{R}} = ((\Phi_{\sigma\, r})_{r\in\mathcal{S}_\mathrm{R}})^{\smile} = (\Phi\lfloor\sigma)^{\smile}$$

Preservation of meet:

$$\begin{array}{rcl}(\Phi \sqcap^\Sigma \Psi)\lfloor\sigma & = & ((\Phi \sqcap \Psi)_{\sigma\, r})_{r\in\mathcal{S}_\mathrm{R}} \;=\; (\Phi_{\sigma\, r} \sqcap \Psi_{\sigma\, r})_{r\in\mathcal{S}_\mathrm{R}} \\ & = & (\Phi_{\sigma\, r})_{r\in\mathcal{S}_\mathrm{R}} \sqcap (\Psi_{\sigma\, r})_{r\in\mathcal{S}_\mathrm{R}} \;=\; (\Phi\lfloor\sigma) \sqcap^{\Sigma_\mathrm{R}} (\Psi\lfloor\sigma)\end{array}$$ □

Joins that are defined component-wise are preserved in the same way.

Obviously, the reduct relator is in general not full if σ is not injective on sorts.

If σ is injective, we can "replace in $\mathcal{A}$ its reduct part along a morphism to $\mathcal{A}\lfloor\sigma$", which will be useful in the next section:

Theorem 7.4 If $\sigma : \Sigma_\mathrm{R} \to \Sigma$ is an injective signature homomorphism, then the reduct functor $\lfloor\sigma$ is a fibration [BW99, 12.1].

PROOF: If $\mathcal{A}$ is an object in $\mathbf{C}^\Sigma$, $\mathcal{R}$ is an object in $\mathbf{C}^{\Sigma_\mathrm{R}}$, and $\phi : \mathcal{R} \to \mathcal{A}\lfloor\sigma$ is a morphism in $\mathbf{C}^{\Sigma_\mathrm{R}}$, then we construct an object $\mathcal{B}$ in $\mathbf{C}^\Sigma$ and a morphism $\psi : \mathcal{B} \to \mathcal{A}$ as follows:

- For every $s : \mathcal{S}$ outside the range of σ, we let $s^\mathcal{B} := s^\mathcal{A}$ and $\psi_s := \mathbb{I}_{s^\mathcal{A}}$.
- For every $r : \mathcal{S}_\mathrm{R}$, we let $(\sigma\, r)^\mathcal{B} := r^\mathcal{R}$ and $\psi_{\sigma\, r} := \phi_r$.
- For every $f : \mathcal{F}$ outside the range of σ, we let $f^\mathcal{B} := \psi_{\mathsf{src} f} \mathbin{;} f^\mathcal{A} \mathbin{;} \psi^{\smile}_{\mathsf{trg} f}$.
- For every $g : \mathcal{F}_\mathrm{R}$, we let $(\sigma\, g)^\mathcal{B} := \phi^{\smile}_{\mathsf{src} g} \mathbin{;} g^\mathcal{R} \mathbin{;} \phi_{\mathsf{trg} g}$.

Well-definedness is easily verified. We can now show that ψ is *cartesian for ψ and $\mathcal{A}$*:

If $v : \mathcal{Z} \to \mathcal{A}$ in $\mathbf{C}^\Sigma$ and $h : \mathcal{Z}\lfloor\sigma \to \mathcal{R}$ such that $h \mathbin{;} \phi = v\lfloor\sigma$, then $w : \mathcal{Z} \to \mathcal{B}$ defined by

- for every $s : \mathcal{S}$ outside the range of σ, let $w_s := v_s$,
- for every $r : \mathcal{S}_\mathrm{R}$, let $w_{\sigma\, r} := h_r$,

obviously satisfies $w \mathbin{;} \psi = v$ and $w\lfloor\sigma = h$, and obviously is the unique such arrow. □

8 Reduct-Restricted Σ-Algebra Categories

In the following, let $\sigma : \Sigma_\mathrm{R} \to \Sigma$ be an arbitrary but fixed signature homomorphism, and $\mathbf{K}$ a sub-category of $\mathbf{C}^{\Sigma_\mathrm{R}}$. We will further assume that $\mathbf{K}$ is contained in the image of $\lfloor\sigma$ — this restriction is not essential, but frequently allows more concise formulations.

Definition 8.1 The $\sigma,\mathbf{K}$-restriction of $\mathbf{C}^\Sigma$ contains exactly those objects and morphisms for which the image under $\lfloor\sigma$ is in $\mathbf{K}$.

We denote this restriction as $\mathbf{C}^{\sigma|_\mathbf{K}}$. □

Because relators preserve identities and composition, and $\mathbf{K}$ is a category, the restriction $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is a category again.

The technical importance of the assumption on $\mathbf{K}$ is that it provides surjectivity on homsets for the reduct relator:

Proposition 8.2 If $\mathbf{K}$ is contained in the image of $\lfloor\sigma$, then the restriction of $\lfloor\sigma$ to $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is a full relator. □

If σ is a sub-signature embedding, we also write $\mathbf{C}^{\Sigma|_{\mathbf{K}}}$ instead of $\mathbf{C}^{\sigma|_{\mathbf{K}}}$. If, in addition, the restriction category $\mathbf{K}$ contains only one object $\mathcal{L}$ and its identity, we also write $\mathbf{C}^{\Sigma|_{\mathcal{L}}}$.

This latter case covers in particular the situation where Σ_{R} contains only label sorts and $\mathcal{L}$ fixes the label interpretations, producing for example a category of labelled graphs with fixed label sets.

Note that every one-object-one-morphism category has all limits and colimits and is not only an allegory, but even a (trivial) relation algebra, and also a bi-tabular collagory. This therefore provides an important special case for many of the properties in the remainder of this paper.

Proposition 8.3 If $\mathbf{K}$ is a sub-allegory of $\mathbf{C}^{\Sigma_{\mathrm{R}}}$, then $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is an allegory.

Proof: Assume that $\Phi\lfloor\sigma$ and $\Psi\lfloor\sigma$ are in $\mathbf{K}$. Since $\mathbf{K}$ is closed under converse and meets, $\Phi^{\smile}\lfloor\sigma = (\Phi\lfloor\sigma)^{\smile}$ and $(\Phi \sqcap \Psi)\lfloor\sigma = (\Phi\lfloor\sigma) \sqcap (\Psi\lfloor\sigma)$ are in $\mathbf{K}$, too.

Therefore, $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is closed under converse and meets, too, and therefore is a sub-allegory of $\mathbf{C}^{\Sigma}$. □

Proposition 8.4 For semi-unary Σ, if $\mathbf{K}$ is a sub-collagory of $\mathbf{C}^{\Sigma_{\mathrm{R}}}$, then $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is a collagory.

Proof: Assume that $\Phi\lfloor\sigma$ and $\Psi\lfloor\sigma$ are in $\mathbf{K}$. With Lemma 6.5 and since $\mathbf{K}$ is closed under joins, the join $(\Phi \sqcup^{\Sigma} \Psi)\lfloor\sigma = (\Phi\lfloor\sigma) \sqcup^{\Sigma_{\mathrm{R}}} (\Psi\lfloor\sigma)$ is in $\mathbf{K}$, too.

So $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is closed under joins, too, and therefore is a sub-collagory of $\mathbf{C}^{\Sigma}$. □

This join preservation works in particular in the case where $\mathbf{K}$ is a one-object-one-morphism category, since in that case, non-empty joins in $\mathbf{K}$ are still inherited (trivially) from $\mathbf{C}^{\Sigma_{\mathrm{R}}}$.

Empty joins, i.e., least morphisms, however, are generally *not* inherited in the one-object-one-morphism category, since identity morphisms are rarely least morphisms in $\mathbf{C}^{\Sigma_{\mathrm{R}}}$. Therefore the zero law does in general not hold in $\mathbf{C}^{\sigma|_{\mathbf{K}}}$. A simple example for this arises in $Set^{\mathsf{sigPointed}|_{\{\bullet\}}}$, i.e., the allegory of relational homomorphisms between pointed sets: The presence of the point induces exactly the same counterexamples as the presence of a zero-ary function symbol, for example if $\mathsf{O}^{\mathcal{A}} = \{0,1\}$, and the point (respectively the value of the constant) in $\mathcal{A}$ is 1, then $\perp\!\!\!\perp_{\mathsf{O}^{\mathcal{A}},\mathsf{O}^{\mathcal{A}}} = \{(1,1)\}$ is a non-trivial closure of the non-inherited least element of $\mathbf{K}$, and with $R := \{(0,1),(1,1)\}$ we have $R ; \perp\!\!\!\perp = R \neq \perp\!\!\!\perp$.

Since the reduct relator $\lfloor\sigma$ distributes over all relevant operations, it also preserves tabulations and co-tabulations, i.e.:

- If the span $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ is a tabulation for the morphism $V : \mathcal{B} \to \mathcal{C}$ in $\mathbf{C}^{\Sigma}$, then the span $\mathcal{B}\lfloor\sigma \xleftarrow{P\lfloor\sigma} \mathcal{A}\lfloor\sigma \xrightarrow{Q\lfloor\sigma} \mathcal{C}\lfloor\sigma$ is a tabulation for $(V\lfloor\sigma) : \mathcal{B}\lfloor\sigma \to \mathcal{C}\lfloor\sigma$ in $\mathbf{C}^{\Sigma_{\mathrm{R}}}$.

- If the co-span $\mathcal{B} \xrightarrow{R} \mathcal{D} \xleftarrow{S} \mathcal{C}$ is a co-tabulation for the difunctional morphism $W : \mathcal{B} \to \mathcal{C}$ in $\mathbf{C}^{\Sigma}$, then the co-span $\mathcal{B}\lfloor\sigma \xrightarrow{R\lfloor\sigma} \mathcal{D}\lfloor\sigma \xleftarrow{S\lfloor\sigma} \mathcal{C}\lfloor\sigma$ is a co-tabulation for $(W\lfloor\sigma) : \mathcal{B}\lfloor\sigma \to \mathcal{C}\lfloor\sigma$ in $\mathbf{C}^{\Sigma_{\mathrm{R}}}$.

Theorem 8.5 For semi-unary Σ, if $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$ is injective, $\mathbf{K}$ is a sub-collagory of $\mathbf{C}^{\Sigma_{\mathrm{R}}}$, the morphism $V : \mathcal{B} \to \mathcal{C}$ has a tabulation $\mathcal{B} \xleftarrow{P} \mathcal{A} \xrightarrow{Q} \mathcal{C}$ in $\mathbf{C}^{\Sigma}$, and $V\lfloor\sigma$ has a tabulation $\mathcal{B}\lfloor\sigma \xleftarrow{P_0} \mathcal{A}_0 \xrightarrow{Q_0} \mathcal{C}\lfloor\sigma$ in $\mathbf{K}$, then V also has a tabulation in $\mathbf{C}^{\sigma|_{\mathbf{K}}}$.

PROOF: Since tabulations in $\mathbf{C}^{\Sigma_{\mathrm{R}}}$ are unique up to isomorphism, there must be an isomorphism $\phi : \mathcal{A}_0 \to \mathcal{A}\lfloor\sigma$. According to Theorem 7.4, we obtain a cartesian morphism $\psi : \mathcal{A}_1 \to \mathcal{A}$ for ϕ and $\mathcal{A}$, and since this is also an isomorphism, $\mathcal{B} \xleftarrow{\psi;P} \mathcal{A}_1 \xrightarrow{\psi;Q} \mathcal{C}$ is a tabulation for V in $\mathbf{C}^{\sigma|_{\mathbf{K}}}$. □

The corresponding statement for co-tabulations is shown in the same way, so we obtain as result:

Theorem 8.6 For semi-unary Σ and an injective signature homomorphism $\sigma : \Sigma_{\mathrm{R}} \to \Sigma$, if $\mathbf{C}$ is a bi-tabular collagory and if $\mathbf{K}$ is bi-tabular sub-collagory of $\mathbf{C}^{\Sigma_{\mathrm{R}}}$, then $\mathbf{C}^{\sigma|_{\mathbf{K}}}$ is a bi-tabular collagory, too. □

This includes all the systematically constructed examples for adhesive categories provided by Lack and Sobociński [LS04], in particular the following uses of a one-object-one-morphism collagory $\mathbf{K}$:

Corollary 8.7 If $\mathbf{C}$ is a bi-tabular collagory, then the following are bi-tabular collagories, too:

- $\mathbf{C}^{\mathsf{sigPointed}|_{\mathcal{C}}}$ for any object $\mathcal{C}$ (conflating $\mathcal{C}$ in $\mathbf{C}$ with the sigPoint-algebra that assigns $\mathcal{C}$ to the sort P) — this is equivalent to the co-slice category $\mathcal{C}/\mathbf{C}$,
- $\mathbf{C}^{\mathsf{sigTyped}|_{\mathcal{C}}}$ for any object — this is equivalent to the slice category $\mathbf{C}/\mathcal{C}$,
- node- and edge-labelled graphs considered as sigLGraph-algebras with fixed node and edge label sets. □

9 Conclusion

We have streamlined the axiomatic basis of the relation-algebraic approach to graph structure transformation by introducing *collagories*, which, in comparison to earlier approaches, remove consideration of the zero-law and, to a certain extent, of difunctional closure defined via the Kleene star. We showed that the concepts of tabulation and co-tabulation, which are essential for the relation-algebraic rewriting approach, can be formalised in collagories, and that the category of mappings in a bi-tabular collagory forms an *adhesive category*, thus establishing a powerful connection to the categorical approach to graph structure transformation. We showed that all the important examples of adhesive categories can also be obtained as special cases of powerful collagory constructions; future work will investigate whether (respectively when) the category of relations [FS90, 1.412] in an adhesive category forms a collagory. Another interesting goal would be to identify a nicer collagory-level formulation of the van Kampen property, and establish connections with the characterisation as bicolimits in the bicategory of spans given by Heindel and Sobociński [HS09].

Further investigations will explore different variations of adhesive categories in a collagory setting, including the quasiadhesive categories of [LS05], and their applications.

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